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Bernoulli Polynomials and Convolution Sums

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Abstract

In this paper, we study the summations $\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+c=n}} (f(a+b)+f(b+c)+f(c+a))$ and $\sum_{\substack{(a,b,c,d)\in\mathbb{N}^4\\ a+b+c+d=n}} (f(a+b+c)+f(b+c+d)+f(c+d+a))$. Especially, for the first summation we obtain some relations related to Bernoulli and Euler polynomials.

Keywords: Convolution sum; Bernoulli polynomials

2010 Mathematics Subject Classification: 11A05

1 Introduction

The study of arithmetical identities is classical in number theory and such investigations have been carried out by several mathematicians including the legend Srinivasa Ramanujan. For $n \in \mathbb{N}$ with $l \in \mathbb{N} \cup \{0\}$, we define some notations which also appear in many areas of number theory:

$$\sigma_l(n) := \sum_{d|n} d^l, \quad S_l(n) = \sum_{i=1}^{n-1} i^l \quad (n \neq 1), \quad S_l(1) = 0,$$

and $[x]$ called the *floor* of x, denotes the greatest integer that does not exceed x.

The Bernoulli polynomials $B_m(x)$, which are usually defined by the exponential generating function

$$\frac{te^{xt}}{e^t-1} = \sum_{m=0}^{\infty} B_m(x)\frac{t^m}{m!},$$

play an important role in various areas of mathematics, including number theory and the theory of finite differences (see [2]). The Bernoulli polynomials satisfy the following well-known identities :

$$B_m(x+1) - B_m(x) = mx^{m-1} \quad (m \in \mathbb{N})$$

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and

$$\sum_{j=0}^{n} j^m = \frac{B_{m+1}(n+1) - B_{m+1}(0)}{m+1} \quad (m \in \mathbb{N}), \tag{1.1}$$

where $\mathbb{N}$ denotes the set of positive integers. We set $B_m := B_m(0)$. It is obvious, from the way the polynomials $B_m(x)$ are constructed, that all the B_m are rational numbers. It can be shown that $B_{2m+1} = 0$ for $m \geq 1$, and is alternatively positive and negative for even m.

Let n be a positive integer and let $f : \mathbb{Z} \to \mathbb{C}$ be an even function. In 1858 Liouville stated the surprising result that the sum called the third identity of Liouvile

$$\begin{aligned}&\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} (f(a-b) - f(a+b))\\ &= f(0)\left(\sigma_1(n) - \sigma_0(n)\right) + \sum_{\substack{d\in\mathbb{N}\\ d\mid n}} \left(1 + \frac{2n}{d} - d\right) f(d) - 2\sum_{\substack{d\in\mathbb{N}\\ d\mid n}} \left(\sum_{v=1}^{d} f(v)\right)\end{aligned}$$

can be evaluated in terms of sums over the positive integers d dividing n (see ([5], p. 247)). In 2000 Huard, Ou, Spearman and Williams (see [7]) proved a far reaching generalization of Liouville's Identity as Proposition 1.1. We introduce Proposition 1.1 which is the motivation of our results Lemma 2.1 and Lemma 2.3.

Proposition 1.1. *([7], Theorem 13.1) Let $f : \mathbb{Z}^4 \to \mathbb{C}$ be such that*

$$f(a,b,x,y) - f(x,y,a,b) = f(-a,-b,x,y) - f(x,y,-a,-b) \tag{1.2}$$

for all integers a, b, x and y. Let $n \in \mathbb{N}$. Then

$$\begin{aligned}&\sum_{\substack{(a,b,x,y)\in\mathbb{N}^4\\ ax+by=n}} (f(a,b,x,-y) - f(a,-b,x,y) + f(a,a-b,x+y,y)\\ &\qquad - f(a,a+b,y-x,y) + f(b-a,b,x,x+y) - f(a+b,b,x,x-y))\\ &= \sum_{\substack{d\in\mathbb{N}\\ d\mid n}} \sum_{\substack{x\in\mathbb{N}\\ x<d}} (f(0,\frac{n}{d},x,d) + f(\frac{n}{d},0,d,x) + f(\frac{n}{d},\frac{n}{d},d-x,-x)\\ &\qquad - f(x,x-d,\frac{n}{d},\frac{n}{d}) - f(x,d,0,\frac{n}{d}) - f(d,x,\frac{n}{d},0)),\end{aligned} \tag{1.3}$$

where the sum on the left hand side of (1.3) is over all positive integers a, b, x, y satisfying $ax + by = n$, the inner sum on the right hand side is over all positive integers x satisfying $x < d$, and the outer sum on the right hand side is over all positive integers d dividing n. (See also ([3], Theorem 1))

In this paper we will find a special formula of the summation

$$\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+c=n}} (f(a+b) + f(b+c) + f(c+a))$$

and so, by using this, we will present some relations related to Bernoulli polynomial and the generalized Riemann zeta function :

Theorem 1.1. *Let $n \in \mathbb{N}$. If $l \in \mathbb{N}$, then we have*

$$\begin{aligned} A(l,n) := & \sum_{i=0}^{l} \binom{l}{i} \sum_{j=0}^{i} \binom{i}{j} \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^{m+j} \\ & \times \binom{l+1-i+j}{m} B_m \sum_{k=1}^{n-2} (n-k)^{i-j} (n-k-1)^{l+1-i+j-m} \\ = & \frac{(l+1)\left(B_{l+2}(n) - B_{l+2}\right) - (l+2)\left(B_{l+1}(n) - B_{l+1}\right)}{(l+2)(l+1)}. \end{aligned}$$

Furthermore, if $l \in \mathbb{N} \cup \{0\}$, then we obtain

$$A(l,n) = S_{l+1}(n) - S_l(n).$$

Corollary 1.2. *Let $t, l \in \mathbb{N} \cup \{0\}$. Then we have*

(a)

$$\sum_{l=0}^{t} A(l,n) = S_{t+1}(n) - S_0(n).$$

(b) *Let $\overline{A(l,n)} = S_{l+1}(n) + S_l(n)$. So*

$$\sum_{l=0}^{t} A(l,n)\overline{A(l,n)} = S_{t+1}^2(n) - S_0^2(n).$$

Corollary 1.3. *Let $l, n \in \mathbb{N}$. Then we have*

$$\begin{aligned} & 2^{l+1} A\left(l, \left[\frac{n+1}{2}\right]\right) - A(l,n) + 2^{l+1} S_{l+1}\left(\left[\frac{n+1}{2}\right]\right) \\ = & \frac{E_{l+1}(0) - E_l(0) - (-1)^n \left(E_{l+1}(n) - E_l(n)\right)}{2}, \end{aligned}$$

where Euler polynomials $E_l(x)$ are defined by $\sum_{l=0}^{\infty} E_l(x) \frac{t^l}{l!} = \frac{2e^{xt}}{e^t+1}$.

2 Proofs of Lemma 2.1, Theorem 1.1 and Corollary 1.2

Using the four basic theta functions, one is led to the formula (see ([7], p. 14)) :

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=2n \\ a,b,x,y \text{ odd}}} \left((a+b)^{2k} - (a-b)^{2k}\right) = \sum_{\substack{m \in \mathbb{N} \\ m|n \\ \frac{n}{m} \text{ odd}}} 2^{2k} m^{2k+1}, \quad (k,n \in \mathbb{N}). \tag{2.1}$$

There are several arithmetical formulae due to Liouville. A classical one of Liouville very often used in elementary number theory (completely avoiding the hard analysis) is that (see ([4], p. 144)) : For $n \in \mathbb{N}$ and for $f : \mathbb{Z} \to \mathbb{C}$ an even function, the identity

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4 \\ ax+by=2n \\ a,b,x,y \text{ odd}}} \left(f(a+b) - f(a-b)\right) = \sum_{\substack{m \in \mathbb{N} \\ m|n \\ \frac{n}{m} \text{ odd}}} \left(f(2m) - f(0)\right) \tag{2.2}$$

holds.

Lemma 2.1. *Let $n \geq 3$ be a positive integer. Let $f : \mathbb{Z} \to \mathbb{C}$ be a function. Then*

$$\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\a+b+c=n}} (f(a+b)+f(b+c)+f(c+a)) = 3\sum_{k=1}^{n-2}(n-k-1)\,f(n-k).$$

Proof. We note that

$$\begin{aligned}
&\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\a+b+c=n}} (f(a+b)+f(b+c)+f(c+a))\\
&= 3\sum_{a+b+c=n} f(a+b) = 3\sum_{k\geq 2} f(k) \sum_{\substack{a+b+c=n\\a+b=k}} 1\\
&= 3\left\{(n-2)f(n-1)+(n-3)f(n-2)+\cdots+(n-(n-1))\,f(2)\right\}\\
&= 3\sum_{k=1}^{n-2}(n-k-1)\,f(n-k).
\end{aligned}$$

□

We can see the arithmetic function $F_l(n)$ given by

$$F_l(n) := \begin{cases} 1, & l \mid n,\\ 0, & l \nmid n \end{cases}$$

for $l, n \in \mathbb{N}$ in ([7], p. 32). Using this function we introduce Example 2.2 and Example 2.4.

Example 2.2. *Let us consider the function $f(x) = F_2(x)$ in* Lemma 2.1. If n is even, then the left hand side of Lemma 2.1 is

$$\begin{aligned}
&\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\n\in\mathbb{N}\\a+b+c=2n}} (f(a+b)+f(b+c)+f(c+a))\\
&= 3\sum_{a+b+c=2n} F_2(a+b) = 3\sum_{\substack{a+b+c=2n\\2\mid(a+b)}} 1
\end{aligned}$$

and the right hand side of Lemma 2.1 is

$$\begin{aligned}
&3\sum_{k=1}^{2n-2}(2n-k-1)\,f(2n-k) = 3\sum_{k=1}^{2n-2}(2n-k-1)\,F_2(2n-k)\\
&= 3\sum_{\substack{k=1\\2\mid(2n-k)}}^{2n-2}(2n-k-1) = 3\sum_{\substack{k=1\\2\mid k}}^{2n-2}(2n-k-1)\\
&= 3\sum_{k=1}^{n-1}(2n-2k-1) = 3(n-1)^2,
\end{aligned}$$

where we use $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. These facts show that

$$\sum_{\substack{a+b+c=2n \\ 2|(a+b)}} 1 = (n-1)^2 \tag{2.3}$$

Also, if n is odd, then we obtain

$$\begin{aligned}
&\sum_{\substack{(a,b,c)\in\mathbb{N}^3 \\ n\in\mathbb{N} \\ a+b+c=2n-1}} (f(a+b)+f(b+c)+f(c+a)) \\
&= 3\sum_{a+b+c=2n-1} F_2(a+b) = 3\sum_{\substack{a+b+c=2n-1 \\ 2|(a+b)}} 1 \\
&= 3\sum_{k=1}^{2n-3} (2n-2-k)\, F_2(2n-1-k) = 3\sum_{\substack{k=1 \\ 2|(2n-1-k)}}^{2n-3} (2n-2-k) \qquad (2.4) \\
&= 3\sum_{\substack{k=1 \\ 2\nmid k}}^{2n-3} (2n-2-k) = 3\left\{\sum_{k=1}^{2n-3} (2n-2-k) - \sum_{\substack{k=1 \\ 2|k}}^{2n-3} (2n-2-k)\right\} \\
&= 3\left\{\sum_{k=1}^{2n-3} (2n-2-k) - \sum_{k=1}^{n-2} (2n-2-2k)\right\} = 3(n-1)^2.
\end{aligned}$$

Therefore, by (2.3) and (2.4), we conclude that

$$\sum_{\substack{n\in\mathbb{N} \\ a+b+c=n \\ 2|(a+b)}} 1 = \left(\left[\frac{n+1}{2}\right]-1\right)^2.$$

We can prove Lemma 2.3 in a similar manner as in the proof of Lemma 2.1 as follows :

Lemma 2.3. *Let $n \geq 4$ be a positive integer. Let $f : \mathbb{Z} \to \mathbb{C}$ be a function. Then*

$$\sum_{\substack{(a,b,c,d)\in\mathbb{N}^4 \\ a+b+c+d=n}} (f(a+b+c)+f(b+c+d)+f(c+d+a)) = \frac{3}{2}\sum_{k=1}^{n-1}(k-1)(k-2)f(k).$$

Proof. We observe that

$$\begin{aligned}
&\sum_{\substack{(a,b,c,d)\in\mathbb{N}^4 \\ a+b+c+d=n}} (f(a+b+c)+f(b+c+d)+f(c+d+a)) \\
&= 3\sum_{a+b+c+d=n} f(a+b+c) = 3\sum_{k\geq 3} f(k) \sum_{\substack{a+b+c+d=n \\ a+b+c=k}} 1 \\
&= 3\sum_{k=3}^{n-1} f(k)\,\{1+2+...+(k-2)\} = 3\sum_{k=3}^{n-1} f(k)\cdot\frac{(k-1)(k-2)}{2} \\
&= \frac{3}{2}\sum_{k=1}^{n-1}(k-1)(k-2)f(k).
\end{aligned}$$

□

Example 2.4. *We apply the function* $f(x) = F_2(x)$ *in* Lemma 2.3. If n is even, then the left hand side of Lemma 2.3 is

$$\begin{aligned}
&\sum_{\substack{(a,b,c,d)\in\mathbb{N}^4 \\ n\in\mathbb{N} \\ a+b+c+d=2n}} (f(a+b+c)+f(b+c+d)+f(c+d+a)) \\
&= 3\sum_{a+b+c+d=2n} F_2(a+b+c) = 3\sum_{\substack{a+b+c+d=2n \\ 2|(a+b+c)}} 1
\end{aligned}$$

and the right hand side of Lemma 2.3 is

$$\begin{aligned}
&\frac{3}{2}\sum_{k=1}^{2n-1}(k-1)(k-2)f(k) = \frac{3}{2}\sum_{k=1}^{2n-1}(k-1)(k-2)F_2(k) \\
&= \frac{3}{2}\sum_{\substack{k=1 \\ 2|k}}^{2n-1}(k-1)(k-2) = \frac{3}{2}\sum_{k=1}^{n-1}(2k-1)(2k-2) \\
&= \frac{3}{2}\sum_{k=1}^{n-1}\left\{4k^2-6k+2\right\} = \frac{1}{2}\left\{4n^3-15n^2+17n-6\right\},
\end{aligned}$$

where we use $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$. These facts lead us to

$$\sum_{\substack{a+b+c+d=2n \\ 2|(a+b+c)}} 1 = \frac{1}{6}\left\{4n^3-15n^2+17n-6\right\}. \tag{2.5}$$

Similarly, if n is odd, then we have

$$\begin{aligned}
&\sum_{\substack{(a,b,c,d)\in\mathbb{N}^4 \\ n\in\mathbb{N} \\ a+b+c+d=2n-1}} (f(a+b+c)+f(b+c+d)+f(c+d+a)) \\
&= 3\sum_{a+b+c+d=2n-1} F_2(a+b+c) = 3\sum_{\substack{a+b+c+d=2n-1 \\ 2|(a+b+c)}} 1 \\
&= \frac{3}{2}\sum_{k=1}^{2n-2}(k-1)(k-2)F_2(k) = \frac{3}{2}\sum_{\substack{k=1 \\ 2|k}}^{2n-2}(k-1)(k-2) \\
&= \frac{3}{2}\sum_{k=1}^{n-1}(2k-1)(2k-2) = \frac{1}{2}\left\{4n^3-15n^2+17n-6\right\}.
\end{aligned} \tag{2.6}$$

Therefore, by (2.5) and (2.6), we conclude that

$$\sum_{\substack{n\in\mathbb{N} \\ a+b+c+d=n \\ 2|(a+b+c)}} 1 = \frac{1}{6}\left\{4\left[\frac{n+1}{2}\right]^3 - 15\left[\frac{n+1}{2}\right]^2 + 17\left[\frac{n+1}{2}\right] - 6\right\}.$$

***Proof of* Theorem 1.1**. Let $l \in \mathbb{N} \cup \{0\}$. We apply $f(x) = x^l$ in Lemma 2.1. Then the left hand side is

$$\begin{aligned}
&\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+b+c=n}} (f(a+b)+f(b+c)+f(c+a))\\
&= 3\sum_{a+b+c=n} f(a+b) = 3\sum_{a+b+c=n} (a+b)^l\\
&= 3\sum_{a+b+c=n}\sum_{i=0}^{l}\binom{l}{i}a^{l-i}b^i\\
&= 3\sum_{i=0}^{l}\binom{l}{i}\sum_{a+b+c=n} a^{l-i}b^i c^0.
\end{aligned} \tag{2.7}$$

Here, we can observe that

$$\begin{aligned}
&\sum_{a+b+c=n} a^{l-i}b^i c^0\\
&= \sum_{a+b=n-1} a^{l-i}b^i 1^0 + \sum_{a+b=n-2} a^{l-i}b^i 2^0 + \cdots + \sum_{a+b=2} a^{l-i}b^i (n-2)^0\\
&= \sum_{a+b=n-1} a^{l-i}b^i + \sum_{a+b=n-2} a^{l-i}b^i + \cdots + \sum_{a+b=2} a^{l-i}b^i.
\end{aligned} \tag{2.8}$$

Then we can calculate the first term of (2.8) as follows:

$$\begin{aligned}
&\sum_{a+b=n-1} a^{l-i}b^i = \sum_{a=1}^{n-2} a^{l-i}\{(n-1)-a\}^i\\
&= \sum_{a=1}^{n-2} a^{l-i}\sum_{j=0}^{i}\binom{i}{j}(n-1)^{i-j}(-1)^j a^j\\
&= \sum_{j=0}^{i}\binom{i}{j}(n-1)^{i-j}(-1)^j\sum_{a=1}^{n-2} a^{l-i+j}\\
&= \sum_{j=0}^{i}\binom{i}{j}(n-1)^{i-j}(-1)^j\frac{1}{l+1-i+j}\sum_{m=0}^{l-i+j}(-1)^m\binom{l+1-i+j}{m}\\
&\quad\times B_m(n-2)^{l+1-i+j-m}\\
&= \sum_{j=0}^{i}\binom{i}{j}\frac{1}{l+1-i+j}\sum_{m=0}^{l-i+j}(-1)^{m+j}\binom{l+1-i+j}{m}B_m\\
&\quad\times (n-1)^{i-j}(n-2)^{l+1-i+j-m},
\end{aligned}$$

where we use the Faulhaber's sum (see [1]) as

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1}\sum_{j=0}^{p}(-1)^j\binom{p+1}{j}B_j n^{p+1-j}$$

with the Bernoulli numbers B_j. Also we can apply the same method into the other terms of (2.8). So (2.8) becomes

$$
\begin{aligned}
&\sum_{a+b+c=n} a^{l-i} b^i c^0 \\
&= \sum_{j=0}^{i} \binom{i}{j} \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^{m+j} \binom{l+1-i+j}{m} B_m \\
&\quad \times \Big\{ (n-1)^{i-j} (n-2)^{l+1-i+j-m} + (n-2)^{i-j} (n-3)^{l+1-i+j-m} \\
&\quad + \cdots + (n-(n-2))^{i-j} (n-(n-1))^{l+1-i+j-m} \Big\} \\
&= \sum_{j=0}^{i} \binom{i}{j} \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^{m+j} \binom{l+1-i+j}{m} B_m \\
&\quad \times \sum_{k=1}^{n-2} (n-k)^{i-j} (n-k-1)^{l+1-i+j-m}.
\end{aligned}
$$

So (2.7) can be written as

$$
\begin{aligned}
&\sum_{\substack{(a,b,c)\in\mathbb{N}^3 \\ a+b+c=n}} (f(a+b) + f(b+c) + f(c+a)) \\
&= 3 \sum_{i=0}^{l} \binom{l}{i} \sum_{j=0}^{i} \binom{i}{j} \frac{1}{l+1-i+j} \sum_{m=0}^{l-i+j} (-1)^{m+j} \\
&\quad \times \binom{l+1-i+j}{m} B_m \sum_{k=1}^{n-2} (n-k)^{i-j} (n-k-1)^{l+1-i+j-m}.
\end{aligned} \tag{2.9}
$$

On the other hand, the right hand side of Lemma 2.1 is

$$
\begin{aligned}
&3 \sum_{k=1}^{n-2} (n-k-1)(n-k)^l \\
&= 3 \sum_{k=1}^{n-2} \Big\{ (n-k)^{l+1} - (n-k)^l \Big\} \\
&= 3 \Big[\Big\{ (n-1)^{l+1} - (n-1)^l \Big\} + \Big\{ (n-2)^{l+1} - (n-2)^l \Big\} \\
&\quad + \cdots + \Big\{ 3^{l+1} - 3^l \Big\} + \Big\{ 2^{l+1} - 2^l \Big\} \Big] \\
&= 3 \left\{ \sum_{k=1}^{n-1} k^{l+1} - \sum_{k=1}^{n-1} k^l \right\}.
\end{aligned} \tag{2.10}
$$

Equating (2.9) and (2.10), we obtain

$$\begin{aligned}A(l,n) &:= \sum_{i=0}^{l}\binom{l}{i}\sum_{j=0}^{i}\binom{i}{j}\frac{1}{l+1-i+j}\sum_{m=0}^{l-i+j}(-1)^{m+j} \\ &\quad\times\binom{l+1-i+j}{m}B_m\sum_{k=1}^{n-2}(n-k)^{i-j}(n-k-1)^{l+1-i+j-m} \\ &= \sum_{k=1}^{n-1}k^{l+1}-\sum_{k=1}^{n-1}k^{l}.\end{aligned} \tag{2.11}$$

For $l \in \mathbb{N}\cup\{0\}$, we can write $A(l,n) = S_{l+1}(n) - S_l(n)$.
Now, let $l \in \mathbb{N}$. Then, by (1.1) and $B_l = B_l(0)$, the right hand side of (2.11) is

$$\begin{aligned}&\sum_{k=1}^{n-1}k^{l+1}-\sum_{k=1}^{n-1}k^{l} \\ &= \frac{B_{l+2}(n)-B_{l+2}(0)}{l+2}-\frac{B_{l+1}(n)-B_{l+1}(0)}{l+1} \\ &= \frac{(l+1)\left(B_{l+2}(n)-B_{l+2}\right)-(l+2)\left(B_{l+1}(n)-B_{l+1}\right)}{(l+2)(l+1)},\end{aligned} \tag{2.12}$$

so that we have

$$A(l,n) = \frac{(l+1)\left(B_{l+2}(n)-B_{l+2}\right)-(l+2)\left(B_{l+1}(n)-B_{l+1}\right)}{(l+2)(l+1)}.$$

□

Remark 2.1. We recall the Riemann zeta function and the generalized Riemann(or Hurwitz) zeta function are given by $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$ and $\zeta(s,q) = \sum_{k=0}^{\infty}\frac{1}{(k+q)^s}$, respectively. Equation (2.10) can be written as

$$\begin{aligned}&3\sum_{k=1}^{n-2}(n-k-1)(n-k)^{l} \\ &= 3\sum_{k=1}^{n-2}\left\{(n-k)^{l+1}-(n-k)^{l}\right\} \\ &= 3\left\{\zeta(-l,n)-\zeta(-l-1,n)-\zeta(-l)+\zeta(-l-1)\right\}.\end{aligned} \tag{2.13}$$

Indeed, observe that

$$\begin{aligned}&\sum_{k=1}^{n-2}\left\{(n-k)^{l+1}-(n-k)^{l}\right\} \\ &= \left\{1^{l+1}+2^{l+1}+3^{l+1}+\cdots+(n-2)^{l+1}+(n-1)^{l+1}\right\} \\ &\quad-\left\{1^{l}+2^{l}+3^{l}+\cdots+(n-2)^{l}+(n-1)^{l}\right\}\end{aligned}$$

and

$$\begin{aligned}
&\zeta(-l-1)-\zeta(-l-1,n)=\sum_{n=1}^{\infty}n^{l+1}-\sum_{k=0}^{\infty}(k+n)^{l+1}\\
&=\Big\{1^{l+1}+2^{l+1}+\cdots+(n-1)^{l+1}+n^{l+1}+(n+1)^{l+1}+\cdots\Big\}\\
&\quad-\Big\{n^{l+1}+(n+1)^{l+1}+(n+2)^{l+1}+(n+3)^{l+1}+\cdots\Big\}\\
&=1^{l+1}+2^{l+1}+3^{l+1}+\cdots+(n-2)^{l+1}+(n-1)^{l+1}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\zeta(-l)-\zeta(-l,n)=\sum_{n=1}^{\infty}n^{l}-\sum_{k=0}^{\infty}(k+n)^{l}\\
&=1^{l}+2^{l}+3^{l}+\cdots+(n-2)^{l}+(n-1)^{l}.
\end{aligned}$$

Thus we can rewrite (2.11) as

$$A(l,n)=\zeta(-l,n)-\zeta(-l-1,n)-\zeta(-l)+\zeta(-l-1).$$

Example 2.5. *If* $l=3$ *in* Theorem 1.1, then we have

$$A(3,n)=\frac{1}{60}n\left(12n^4-45n^3+50n^2-15n-2\right).$$

***Proof of* Corollary 1.2**. (a) From Theorem 1.1, we obtain

$$\begin{aligned}
&\sum_{l=0}^{t}A(l,n)=\sum_{l=0}^{t}\left\{\sum_{k=1}^{n-1}k^{l+1}-\sum_{k=1}^{n-1}k^{l}\right\}=\sum_{k=1}^{n-1}\sum_{l=0}^{t}\Big\{k^{l+1}-k^{l}\Big\}\\
&=\sum_{k=1}^{n-1}\left\{(k-1)+(k^2-k)+(k^3-k^2)+\cdots+(k^t-k^{t-1})+(k^{t+1}-k^t)\right\}\\
&=\sum_{k=1}^{n-1}\Big\{k^{t+1}-1\Big\}=S_{t+1}(n)-S_0(n).
\end{aligned}$$

(b) By the definition of $\overline{A(l,n)}=S_{l+1}(n)+S_l(n)$, we have

$$\begin{aligned}
&\sum_{l=0}^{t}A(l,n)\overline{A(l,n)}=\sum_{l=0}^{t}\left\{S_{l+1}^2(n)-S_l^2(n)\right\}\\
&=(S_1^2(n)-S_0^2(n))+(S_2^2(n)-S_1^2(n))+(S_3^2(n)-S_2^2(n))+\cdots\\
&\quad+(S_t^2(n)-S_{t-1}^2(n))+(S_{t+1}^2(n)-S_t^2(n))\\
&=S_{t+1}^2(n)-S_0^2(n).
\end{aligned}$$

□

***Proof of* Corollary 1.3**. In ([6], (2.1), Theorem 2.1) we can see that

$$T_l(n) := \sum_{i=0}^{n-1}(-1)^i i^l,$$
$$T_l(n) = \frac{E_l(0) - (-1)^n E_l(n)}{2} = 2^{l+1} S_l\left(\left[\frac{n+1}{2}\right]\right) - S_l(n),$$

for $l, n \in \mathbb{N}$. These facts lead us to

$$\begin{aligned} &T_{l+1}(n) - T_l(n) \\ &= \frac{E_{l+1}(0) - (-1)^n E_{l+1}(n)}{2} - \frac{E_l(0) - (-1)^n E_l(n)}{2} \\ &= \frac{E_{l+1}(0) - E_l(0) - (-1)^n \left(E_{l+1}(n) - E_l(n)\right)}{2}. \end{aligned} \tag{2.14}$$

On the other hand, $T_{l+1}(n) - T_l(n)$ can be rewritten as

$$\begin{aligned} &T_{l+1}(n) - T_l(n) \\ &= \left\{2^{l+2} S_{l+1}\left(\left[\frac{n+1}{2}\right]\right) - S_{l+1}(n)\right\} - \left\{2^{l+1} S_l\left(\left[\frac{n+1}{2}\right]\right) - S_l(n)\right\} \\ &= 2^{l+1}\left\{S_{l+1}\left(\left[\frac{n+1}{2}\right]\right) - S_l\left(\left[\frac{n+1}{2}\right]\right)\right\} + 2^{l+1} S_{l+1}\left(\left[\frac{n+1}{2}\right]\right) \\ &\quad - \{S_{l+1}(n) - S_l(n)\} \\ &= 2^{l+1} A\left(l, \left[\frac{n+1}{2}\right]\right) + 2^{l+1} S_{l+1}\left(\left[\frac{n+1}{2}\right]\right) - A(l, n), \end{aligned} \tag{2.15}$$

where we use Theorem 1.1. Equating (2.14) and (2.15), we obtain the proof.

□

3 Conclusions

In this paper, we study a special case summation $\sum_{\substack{(a,b,c)\in\mathbb{N}^3 \\ a+b+c=n}} (f(a+b) + f(b+c) + f(c+a))$ and we show this summation is related to Bernoulli and Euler polynomials.

Competing Interests

The author declares that no competing interests exist.

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