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TESTING EXPONENTIALITY AGAINST EXPONENTIAL BETTER THAN USED IN LAPLACE TRANSFORM ORDER BASED ON GOODNESS OF FIT APPROACH

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ABSTRACT

In this paper, test statistic for testing exponentiality versus exponential better than used in Laplace transform order based on goodness of fit approach is discussed. Pitman's asymptotic efficiencies of our test are calculated and compared with other tests. The percentiles of this test are tabulated. The powers of the test are estimated for famously used distributions in aging problems. In case of censored data our test is applied and the percentiles are calculated and tabulated. Finally, examples in different fields are used as practical applications for the proposed test.

Keywords: Goodness of fit, censored data, Monte Carlo simulation, EBUL.

1 INTRODUCTION

The notion of ageing is very important in reliability analysis. "No ageing" means that the age of a component has no influence on the rest of the lifetime of the component. Concepts of ageing describe how a component or system improves or deteriorates with age. Many classes of life distributions are defined in the literature according to their ageing properties, see for example Barlow and Proschan (1981), Abouammoh and Ahmad (1992) and El-Batal (2002). An important aspect of such classifications is that the exponential distribution is nearly always a member of each class which have the property of memory less "as good as new" and its have a constant failure rate, these features increased the importance of the exponential distribution. Testing exponentiality against different classes of life distributions has good attention from many researchers. For testing against increasing failure rate (IFR), see Proschan and Pyke (1967), Barlow (1968) and Ahmad (1975), among others. For testing against new better than used (NBU), see Hollander and Proschan (1972), Koul (1977), Kumazawa (1983) and Ahmad (1994). For testing against new better than used in Laplace transform order (NBUL) and exponential better than used (EBU), see Diab et al (2009) and El-Batal (2002). For relationship between certain classes of life distributions see Ravi and Prathibha (2012).

Definition 1.1 A life distribution F, with F (0) = 0, survival function $\overline{F}$ and finite mean μ is said to be EBU if

$$\overline{F}(x+t) \le \overline{F}(t)e^{-\frac{x}{\mu}}, x,t>0, \tag{1.1}$$

or

$$\overline{F}_t \le e^{-\frac{x}{\mu}}, x,t>0.$$

Where $\overline{F}_t = \frac{\overline{F}(x+t)}{\overline{F}(t)}$ represents the survival function related to the random residual life time X_t

Note that, the above definition which is introduced by El-Batal (2002), is motivated by comparing the life length X_t of a component of age t with another new component of life length X which is exponential with the same mean μ Recently a new class of life distribution named exponential better than used in Laplace transform order (EBUL) is introduced by Mahmoud et al (2014) which expand the EBU class.

Definition 1.2 X is said to be EBUL if

$$\int_0^\infty e^{-sx}\,\overline{F}(x+t)dx \le \frac{\mu}{(\mu s+1)}\overline{F}(t) \quad x,t>0, s\ge 0.$$

(1. 2)

Testing exponentiality based on goodness of fit approach against many classes of life distributions was studied by some authors such as Abu-Youssef (2009), Kayid et al (2010), Ismail and Abu- Youssef (2012) and Mahmoud and Rady (2013). In the current study, A goodness of fit approach is used to testing exponentiality versus (EBUL). In Section 2 based on U- statistic our test is developed and its asymptotic properties are studied. In that section, Monte Carlo null distribution critical points are simulated for samples size n =5(5)35,39,40(5)50 and the power estimates are also calculated and tabulated. In Section3, we dealing with right-censored data and selected critical values are tabulated. Finally, in Section 4 we discuss some applications to demonstrate the utility of the proposed test in reliability analysis.

2 Testing Against EBUL Alternatives

In this section a test statistic is constructed to test exponentiality against (EBUL) based on goodness of fit approach. The following lemma is needed.

Lemma 2.1 If X a random variable with distribution function F *and* F *belongs to EBUL class, then*

$$(s\mu+1)(1-\phi(s))\le s(\mu+1)(1-\phi(1)),\quad s\ge 0, \tag{2.1}$$

where $\phi(s)=E\,e^{-sX}=\int_0^\infty e^{-sx}\,dF(x)$

Proof.

Since F is EBUL then,

$$\int_0^\infty e^{-sx}\,\overline{F}(x+t)dx\le\frac{\mu}{(\mu s+1)}\overline{F}(t)\quad x,t>0\,.$$

Consider the following integral

$$\int_0^\infty\int_0^\infty e^{-t}e^{-sx}\overline{F}(x+t)dxdt\le\frac{\mu}{(\mu s+1)}\int_0^\infty e^{-t}\overline{F}(t)dt\,. \tag{2.2}$$

Setting

$$I_1=\int_0^\infty e^{-t}\overline{F}(t)dt=E\int_0^\infty e^{-t}I(X>t)dt$$
$$=E\int_0^X e^{-t}dt=(1-Ee^{-X})$$

it is easy to show that

$$I_1=(1-\phi(1)) \tag{2.3}$$

Setting

$$I_2=\int_0^\infty\int_0^\infty e^{-t}e^{-sx}\,\overline{F}(x+t)dxdt.$$

So I_2 can be put in the following form

$$I_2=\int_0^\infty\int_v^\infty e^{-v}e^{-s(u-v)}\overline{F}(u)dudv$$
$$=\int_0^\infty\int_0^v e^{-u(1-s)}e^{-v}\,\overline{F}(v)dudv$$
$$=\frac{1}{1-s}\left[\int_0^\infty e^{-sv}\,\overline{F}(v)dv-\int_0^\infty e^{-v}\overline{F}(v)dv\right],$$

therefore

$$I_2=\frac{1}{1-s}\left[\frac{1}{s}(1-\phi(s))-(1-\phi(1))\right]. \tag{2.4}$$

Substituting (2.3) and (2.4) into (2.2), we get

$$(s\mu+1)(1-\phi(s))\le s(\mu+1)(1-\phi(1))$$

This completes the proof.

Let $X_1 \ldots X_n$ denote a random sample from a distribution F, we wish to test H_1: F is exponential against H_1: F is EBUL and not exponential. By using the following as a measure of departure from H_o in favor of H_1

$$\zeta(s)=s(\mu+1)(1-\phi(1))-(s\mu+1)(1-\phi(s))$$

note that, under H_o $\zeta(s)=0$, while it is positive under H_1. Define the test statistic $\zeta_n(s)$ as follows

$$\zeta_n(s)=\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n[s(1+X_i)(1-e^{-X_j})-(1+sX_i)(1-e^{-sX_j})]$$

To make the test invarient, let

$$\Delta_n(s)=\frac{\zeta_n(s)}{\overline{X}^2},$$

where $\overline{X}=\sum_{i=1}^n\frac{X_i}{n}$ is the sample mean.

Then

$$\Delta_n(s)=\frac{1}{n^2\overline{X}^2}\sum_i\sum_J\phi(X_i,X_j) \tag{2.5}$$

where

$$\phi(X_i,X_j)=s(1+X_i)(1-e^{-X_j})-(1+sX_i)(1-e^{-sX_j}) \tag{2.6}$$

The following theorem summarizes the asymptotic properties of the test.

Theorem 2.2 As $n\to\infty$,($\Delta_n(s)-\zeta(s)$) is asymptotically normal with mean 0 and variance $\sigma^2(s)?n$, where $\sigma^2(s)$ is given in (2.7). Under H_o, the variance reduces to (2.8).

$$\sigma^2=V\{E[\phi(X_1,X_2)\,|\,X_1]+E[\phi(X_1,X_2)\,|\,X_2]\}.$$

Remember the definition of $\phi(X_i,X_j)$ in (2.6), thus it is not difficult to show that

$$E[\phi(X_1,X_2)\,|\,X_1] = s-1-s\int_0^\infty e^{-x}dF(x)+\int_0^\infty e^{-sx}dF(x)+X_1\int_0^\infty e^{-x}dF(x)-X_1\int_0^\infty e^{-sx}dF(x)$$

Similarly,

$$E[\phi(X_1,X_2)\,|\,X_2] = s-1-e^{-X}+e^{-sX}+e^{-X}\int_0^\infty xdF(x)-e^{-sX}\int_0^\infty xdF(x)$$

Hence

$$\sigma^2(s) = Var\{2s-2-s\int_0^\infty e^{-x}dF(x)+\int_0^\infty e^{-sx}dF(x)+X\int_0^\infty e^{-x}dF(x)$$
$$-X\int_0^\infty e^{-sx}dF(x)-e^{-X}+e^{-sX}+e^{-X}\int_0^\infty xdF(x)-e^{-sX}\int_0^\infty xdF(x)\}. \quad (2.7)$$

Under H_o

$$\sigma_o^2(s)=\frac{s^2(s-1)^2(2s^2+s+2)}{2\,(2s+1)(s+2)(s+1)^2}. \quad (2.8)$$

2.1 Asymptotic efficiency

To decide the quality of this procedure, we compare its Pitman asymptotic efficiencies (PAE) with some other tests in Table (1) for the following alternative distributions.

1) The Weibull distribution:

$\overline{F}_1(x)=e^{-x^{\theta}}, x\geq 0, \theta\geq 1.$

2) The linear failure rate distribution (LFR):

$\overline{F}_2(x)=e^{-x-\frac{\theta}{2}x^2}, x\geq 0, \theta\geq 0.$

3) The Makeham am distribution:

$\overline{F}_3(x)=e^{-x-\theta(x+e^{-x}-1)}, x\geq 0, \theta\geq 0.$

Note that when $\theta = 1$ and o, the $(i),(i)$ and (iii) distributions reduce to the exponential distribution.

The PAE is defined by:

$$PAE\,(\Delta_n(s))=\frac{1}{\sigma_o(s)}\left|\frac{d}{d\theta}\zeta(s)\right|_{\theta\to\theta_o}.$$

This leads to:

$$PAE\,[\Delta_n(0.2), Weibull]=7.34682,$$
$$PAE[\Delta_n(0.2), LFR]=4.19399$$

and

$$PAE\,[\Delta_n(0.2), Makeham]=1.71572.$$

Table (1) Comparison between the PAE of our test and some other tests

Test	Weibull	LFR	Makeham
Kango (1993)	0.132	0.433	0.144
Mugdadi and Ahmad (2005)	0.170	0.408	0.039
Mahmoud and Abdul Alim (2008)	0.405	0.433	0.289
Our test	7.34682	4.19399	1.71572

It is clear that our test has the greatest efficiency in all cases.

2.2 Monte Carlo null distribution critical values

In this subsection Monte Carlo null distribution critical points for our test $\Delta_n(0.2)$ are simulated based on 10000 generated samples from the standard exponential distribution using Mathematica 8 program. Table (2) gives the upper percentile points of $\Delta_n(0.2)$, where n =5(5)35,39,40(5)50.

Table (2) The upper percentile points of $\Delta_n(0.2)$ with 10000 replications

n	90%	95%	99%
5	0.00654827	0.00831552	0.0117857
10	0.00680458	0.00866917	0.0123706
15	0.00724927	0.00918529	0.013069
20	0.00802681	0.0101987	0.0142138
25	0.00890195	0.0111627	0.0159395
30	0.00986511	0.0125176	0.017838
35	0.0111477	0.0139494	0.0194316
39	0.012984	0.0163702	0.0227707
40	0.0133555	0.0165929	0.023059
45	0.0174138	0.0221825	0.0304237
50	0.0273877	0.0331176	0.043624

From Table (2), it is obvious that the critical values are increasing as the samples size increasing and they are increasing as the confidence levels increasing.

2.3 The power of test

In this subsection the power of our test $\Delta_n(0.2)$ will be estimated at significance level α = 0.05 with respect to two alternatives Weibull and linear failure rate (LFR) distributions based on 10000 samples. Table (3) gives the power estimates with parameter $\theta = 2,3$ and 4 at n 10.20 and 30

Table(3)ThepowerestimatesoftheStatistic $\Delta_n(0.2)$

n		Weibull	LFR
10	2	0.9928	0.8524
	3	1.0000	0.9037
	4	1.0000	0.926
20	2	0.9991	0.869
	3	1.0000	0.9282
	4	1.0000	0.9563
30	2	0.9987	0.8353
	3	1.0000	0.9266
	4	1.0000	0.9604

It is obviously that the power estimates increase as the sample size increases for each value of θ .

3 Testing for Censored Data

Here, a test statistic is proposed to test H- versus H_1 with randomly right -censored data. Such a censored data is usually the only information available in a life-testing model or in a clinical study where patients may be lost (censored) before the completion of a study. This experimental situation can formally be modeled as follows. Suppose n objects are put on test, and $X_1, X_2, \ldots, X_n$ denote their true life times. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d) according to a continuous life distribution F. Let $Y_1, Y_2, \ldots, Y_n$ be (i.i.d) according to a continuous life distribution G. Also we assume that X's and Y's are independent. In the randomly right -censored model, the pairs (Z_j, δ_j) , $j = 1,\ldots,n,$ are observed, where Z , $= \min(X_j, Y_j$ and

$$\delta_j = \begin{cases} 1, & \text{f} \quad Z_j = X_j \text{ (j-th observation is uncensored)} \\ 0, & \text{f} \quad Z_j = Y_j \text{ (j-th observation is censored)} \end{cases}$$

Let $Z_{(0)} = 0 < Z \quad < Z \quad < \quad \ldots < Z$ (n) denote the orderd Z's and $\delta_{(j)}$ is δ_j corresponding to Z Using the censored data (Z , δ_j), $j = 1,\ldots,n.$ Kaplan and Meier (1958) proposed the product limit estimator, as follows

$$\bar{F}_n(X) = \prod_{[j:\ Z(j)\le X]} \{(n-j)/(n-j+1)\}^{\delta_{(j)}}, X \in [0, Z_{(n)}]$$

Now, for testing H_0 : $\zeta_c(s) = 0$ against H_1 : $\zeta_c(s) > 0,$ using the randomly right censored data, the following test statistic is proposed

$$\zeta_c(s) = s(\mu+1)(1-\phi(1)) - (s\mu+1)(1-\phi(s))$$

For computational purpose, $\zeta_c(s)$ may be rewritten as

$$\zeta_c(s) = s(\Omega+1)(1-\tau) - (s\Omega+1)(1-\eta)$$

where

$$\eta = \sum_{j=1}^{n} e^{-sZ_{(j)}}[\prod_{p=1}^{j-2} C_p^{\delta(p)} - \prod_{p=1}^{j-1} C_p^{\delta(p)}] \quad \Omega = \sum_{k=1}^{n}[\prod_{m=1}^{k-1} C_m^{\delta(m)}(Z_{(k)} - Z_{(k-1)})]$$

$$\tau = \sum_{j=1}^{n} e^{-Z_{(j)}}[\prod_{p=1}^{j-2} C_p^{\delta(p)} - \prod_{p=1}^{j-1} C_p^{\delta(p)}]$$

and

$$dF_n(Z_j) = \bar{F}_n(Z_{j-1}) - \bar{F}_n(Z_j) \quad c_k = [n-k][n-k+1]^{-1}.$$

To make the test invarient, let

$$\Delta_c(s) = \frac{\zeta_c(s)}{\bar{Z}^2}, \text{where } \bar{Z} = \sum_{i=1}^{n} \frac{Z_{(i)}}{n}. \qquad (3.1)$$

3.1 Monte Carlo null distribution critical values in censored case

In this subsection the monte carlo null distribution critical values of Δ_c at $s = 0.2$ for samples sizes n= 20,25,30,40,50,51,60,70,81 with 10000 replications are simulated from the standard exponential distribution by using Mathematica 8 program. Table (4) gives the upper percentile points of the statistic Δ_c.

Table (4) The Upper Percentile Points of Δ_c

n	90%	95%	99%
20	0.341634	68.1421	166861.0
25	3.47767	1584.12	2.29304 10^7
30	8.33774	6179.51	2.16501 10^9
40	39.4047	421367.0	1.77345 * 10^{12}
50	329.65	4.18467 * 10^7	6.42456 * 10^{15}
51	152.416	6.99417 * 10^7	1.09826 * 10^{17}
60	959.563	2.56878 * 10^9	1.81463 * 10^{20}
70	8822.87	1.19424 * 10^{11}	6.36828 * 10^{23}
81	18134.2	6.13836 * 10^{12}	1.36052 * 10^{27}

Table (4) shows that the critical values increase as the sample size and the confidence level increase.

4 Some Applications

In this section, we apply our test on some real data-sets at 95% confidence level.

4.1 Case of non censored data

In this section two examples are presented considering $s = 0.2.$

Example 4.1.1 *Consider the data in Mahmoud et al (2005) which represent 39 liver cancers patients taken from Elminia cancer center Ministry of Health -- Egypt, which entered in (1999). The ordered life times (in days)*

10	14	14	14	14	14	15	17	18	20
20	20	20	20	23	23	24	26	30	30
31	40	49	51	52	60	61	67	71	74
75	87	96	105	107	107	107	116	150	

In this case, $\Delta_n(0.2) = -0.000260314$ which is less than the critical value in Table (2), then we accept H which states that the data set have exponential property.

Example 4.1.2 *Consider the real data-set given in Grubbs (1971) and have been used in Shapiro (1995).This data set gives the times between arrivals of 25 customers at a facility.*

1.80	2.89	2.93	3.03	3.15
3.43	3.48	3.57	3.85	3.92
3.98	4.06	4.11	4.13	4.16
4.23	4.34	4.37	4.53	4.62
4.65	4.84	4.91	4.99	5.17

Since $\Delta_n(0.2) = -0.000353938$ and this value less than the critical value in Table (2). Then we conclude that this data set have the exponential property.

4.2 Case of censored data

In this section two example are presented considering $s = 0.2.$

Example 4.2.1 *Consider the data in Susarla and Vanryzin (1978). These data represent 81 survival times of patients of melanoma. Of them 46 represent whole life times (non-censored data) and the observed values are:*

13	14	19	19	20	21	23	23	25	26	26	27
27	31	32	34	34	37	38	38	40	46	50	53
54	57	58	59	60	65	65	66	70	85	90	98
102	103	110	118	124	130	136	138	141	234		

The ordered censored observations are:

16	21	44	50	55	67	73	76	80	81	86	93
100	108	114	120	124	125	129	130	132	134	140	147
148	151	152	152	158	181	190	193	194	213	215	

Taking into account the whole set of survival data (both censored and uncensored), and computing the statistic from (3.1).

We get $\Delta_c(0.2) = 4.92229 \times 10^{94}$ which is greater than the critical value of Table (4). Then we accept H_1 which states that the data set have EBUL property.

Example 4.2.2 Consider the data in Mahmoud et al (2005) which represent 51 liver cancers patients taken from Elminia cancer center Ministry of Health -- Egypt, which entered in (1999). Out of these 39 represents non-censored data and the others represents censored data. The ordered life times (in days)

- Non-censored data

10	14	14	14	14	14	15	17	18	20
20	20	20	20	23	23	24	26	30	30
31	40	49	51	52	60	61	67	71	74
75	87	96	105	107	107	107	116	150	

- Censored data

30	30	30	30	30	60	150	150	150	150	150	185

Taking into account the whole set of survival data (both censored and uncensored), and computing the statistic using (3.1).

The value of $\Delta_c(0.2)$ is computed to be $8.22584*10^{69}$ which is greater than the critical value of Table (4). There is enough to accept H_1 which states that the data set have EBUL property.

Conclusion

A goodness of fit approach is used to testing exponentiality versus (EBUL). The Pitman asymptotic efficiency of this test is studied. The upper percentiles and the power of the proposed test are calculated and tabulated. In case of censored data the critical values of this test are calculated and tabulated. Our test is applied to some real data. Finally, the proposed test in the two cases seem to be simple.

REFERENCES

- Abouammoh, A. M. and Ahmed, A.N. (1992). On renewal failure rate classes of life distributions. *Stat. Prob. le* ., 14,211-217.
- Abu- Youssef, S. E. (2009). A goodness of fit approach to monotone variance residual life class of life distributions. *Appl. Math. Sc.*, 3(15), 715-724.
- Ahmed, I. A. (1975). A nonparametric test of the monotonicity of a failure rate function, Comm. Statist., 4, 967-974.
- Ahmad, I.A. (1994). Classes of statistics useful in testing increasing failure rate average and new better than used life distributions. *J. Statist. Plan. Inf.*, 41, 141-149.
- Barlow, R. E. (1968). Likelihood ratio tests for restricted families of probability distributions, Ann. Math. Statist. , 39, 547-560.
- Barlow, R. E. and Proschan, F. (1981). Statistical Theory of Reliability and Life Testing. To Begin with Silver Spring, M D.
- Diab, L. S., kayid, M. and Mahmoud, M. A. W. (2009). Moments inequalities for NBUL distribution with hypotheses testing applications, *Contemporary Engineering Sciences*, 2, 319-332.
- El-Batal, I. I. (2002). The EBU and EWU classes of life distribution.J. Egypt Statist. Soc., 18, (1), 59-80.
- Grubbs, F. E. (1971). Fiducial bounds on reliability for the two parameter negative exponential distribution. *Technometrics*, 13, 873-876.
- Hollander, M. and Proschan, F. (1972). Testing whether new is better than used. *Ann. Math. Statist*., 43, 1136-1146.
- Ismail, A. A. and Abu- Youssef, S. E. (2012). A goodness of fit approach to the class of life distribution with unknown age. *Qual. Rel. Eng. Int.* , 25, (7), 761-766.
- Kango, A.I. (1993). Testing for new is better than used, *Comm. Stat. Theor.Meth.*, 12, 311-321.
- Kaplan, E. L. and Meier, P. (1958). Nonparametric estimation from incomplete observation. *A. S. A.*, 53, 457-481.
- Kayid, M., Diab, L. S. and Alzughaibi, A. (2010). Testing NBU (2) class of life distribution based on goodness of fit approach. Journal of King Saud University - Science 22 (4), 241-245.
- Koul, H.L. (1977). A testing for new better than used. *Comm. Statist. Theor. Meth.*, 6, 563-573.
- Kumazawa, Y. (1983). A class of tests statistics for testing whether new is better than used. *Comm. Statist. Theor. Meth.,* 12, 311-321.
- Lee, A.J. (1989). U-Statistics, Marcel Dekker, New York.
- Mahmoud, M. A. W. and Abdul Alim, A. N. (2008). A Goodness of Fit Approach to for Testing NBUFR (NWUFR) and NBAFR (NWAFR) Properties, *Int. J. Reliab. Appl.,* 9, 125-140.
- Mahmoud, M. A. W., Attia, F. A. and Taib, I. B. (2005). Test for Exponential Better than Used in Average Based on The Total Time on Test Transform. The Proceeding of The 17th Annual Conference on Statistics, Modelling in Human and Social Science 29-30, March, 2005, Faculty of Economics and Political Science, Cairo University, Cairo, Egypt.
- Mahmoud, M. A. W. and Rady, D. M. (2013). Testing exponentiality against HNBUET class based on goodness of fit approach. *Al- Azhar Bull. Sci.*, 24, (1), 81-92.
- Mahmoud, M. A. W., Moshref, M. E. and Mansour, M. M. M. (2014). A New Class of Life Distributions with Hypotheses Testing Applications, *Submitted* .
- Mugdadi, A. R. and Ahmad, I. A. (2005). Moment inequalities derived from comparing life with its equlibrium form, *J. Stat. Plann. Inf.*, 134, 303-317.
- Proschan, F. and Pyke, R. (1967). Tests for monotone failure rate, Proc. 5th Berekly Symp., 3293-3312.
- Ravi, S. and Prathibha (2012). Relationship between certain classes of life distributions and some stochastic orderings. *ProbStat Forum*, 5, 1-7.
- Shapiro, S. S. (1995). In The Exponential Distribution Theory Methods and Applications. Balakrishnan, N. and Basu, A. P. Editors, Gorddon and Breach, Amsterdam.
- Susarla, V. and Vanryzin, J. (1978). Empirical Bayes estimations of a survival function right censored observation. *Ann. Stat.*, 6, 710-755.