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Stability of m-Jensen Functional Equations

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Authors' contributions

This work was carried out in collaboration between all authors. Authors TL and NM came out with the idea of setting the problem and performed the functional analysis. Author TL wrote the protocol and the first draft of the manuscript. Authors RQ and ER managed the analysis and the study of stability and research in the literature available. All authors read and approved the final manuscript.

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Abstract

In this research we set a functional equation of Jensen type which comes from the so called Jensen m-convex inequality. We formulate and prove properties of its solutions and give a characterization of them under certain conditions as well. To close the article we have added a section on stability of this type of functional equation, which initially contains some results based on the work from Z. Kominek and finally a result supported on ideas from S. Jung.

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1 Introduction

The beginning of a modern theory of functional equations is connected with the work of an excellent specialist in this field, the Hungarian mathematician J. Aczél. In his numerous papers he treats whole classes of functional equations, gives general methods for solving them and criteria on the existence and uniqueness of solutions. He also indicates new applications of this important topic [1, 2, 3, 4, 5].

Definition 1.1. ([6]) A functional equation is an equality $T_1 = T_2$ between two terms T_1 and T_2 which contain at least one unknown function and a finite number of independent variables. This equality is to be satisfied identically with respect to all occurring variables in a certain set (of any sort).

The solution of a functional equation may depend on the set in which the equation is postulated. One should also precisely state in what function class the solution is sought. The number and behavior of solutions depend on this class. It is one of the important differences between differential and functional equations [6, 7].

In this work, we investigate properties of solutions of a particular functional equation which we call a Jensen type functional equation involving the so called Jensen m-convex functions. This work follows ideas from [7] for the case of Jensen functional equation. For a convex set $D \subseteq \mathbb{R}^N$, a function $f : D \to \mathbb{R}$ is called Jensen convex if it satisfies the so called Jensen inequality,

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

for any $x, y \in D$ [8, 9].

The Jensen equation is the resulting on replacing in the Jensen inequality, mentioned above, the sign of inequality by that of equality [10, 7]

$$f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}. \tag{1.1}$$

Usually, this equation is considered for functions $f : D \to \mathbb{R}$ and D being a convex set.

Definition 1.2. ([11]) A set $D \subseteq \mathbb{R}^N$ is denominated m-convex if for any $x, y \in D$ and $t \in [0, 1]$, the point $tx + m(1-t)y$ also belongs to D.

Definition 1.3. ([12, 13]) Let $m \in (0, 1]$, $c_m = 1 + 1/m$ and $D \subseteq \mathbb{R}^N$ an m-convex set. A function $f : D \to \mathbb{R}$ which satisfies the inequality

$$f\left(\frac{x+y}{c_m}\right) \leq \frac{f(x)+f(y)}{c_m} \text{ for all } x, y \in D \tag{1.2}$$

will be called Jensen m-convex on D.

It is worth to mention that in case $m = 1$, inequality (1.2) becomes the usual Jensen inequality.

In this context the definition of Jensen inequality may be formulated. The functional equation we shall be dealing with is (resulting on replacing in (1.2) the sign of inequality by that of equality)

$$f\left(\frac{x+y}{c_m}\right) = \frac{f(x)+f(y)}{c_m}, \tag{1.3}$$

$x, y \in D$, $D \subseteq \mathbb{R}^N$, m-convex and unless other thing is stated, $0 \in int(D)$ (the interior of set D). Notice that $m = 1$ reduces (1.3) to the usual Jensen equation (1.1).

2 Main Results

Here we set a number of properties of solutions of (1.3) which will drive us to give a characterization of such solutions. From now on, we shall assume $0 < m < 1$.

We will be using the set

$$\frac{1}{c_m^n} D = \left\{ \frac{1}{c_m^n} x : x \in D \right\}$$

for D as before and for any positive integer n.

Before going any further, first, observe that the set $\frac{1}{c_m^n} D$ naturally inherits the m-convexity from D and second, by taking into account that $\frac{m}{m+1} \in [0, 1]$ and the following relationships

$$\frac{1}{c_m} x = \frac{x+0}{c_m} = \frac{m}{m+1} x + m \left(1 - \frac{m}{m+1} \right) 0,$$
$$\frac{1}{c_m^{n+1}} x = \frac{1}{c_m} \left(\frac{1}{c_m^n} x \right)$$

for any $x \in D$ and for any n follows, by induction on n, that

$$\frac{1}{c_m^n} D \subseteq D.$$

The following results are similar to those given in [7] for the Jensen functional equation.

Lemma 2.1. *Let us assume D as before, $f : D \to \mathbb{R}$ be a solution of equation (1.3) then, $f(0) = 0$ and*

$$f\left(\frac{1}{c_m^n} x \right) = \frac{1}{c_m^n} f(x) \tag{2.1}$$

for any $x \in D$ and for any positive integer n.

Proof. We show first $f(0) = 0$, in fact because of the hypothesis and since $0 \in D$,

$$f(0) = f\left(\frac{0+0}{c_m} \right) = \frac{f(0) + f(0)}{c_m} = \frac{2f(0)}{c_m},$$

thus

$$(c_m - 2) f(0) = 0 \text{ and } c_m > 2.$$

For the second part, we use the induction method on n. For $n = 1$, we employ the fact that $f(0) = 0$ and equality (2.1) is easy verified. Assume now (2.1) true for n and check it for $n + 1$.

$$f\left(\frac{1}{c_m^{n+1}} x \right) = f\left(\frac{1}{c_m^n} \left(\frac{1}{c_m} x \right) \right) = \frac{1}{c_m^n} f\left(\frac{1}{c_m} x \right) = \frac{1}{c_m^{n+1}} f(x).$$

Here we use the fact that $\frac{1}{c_m} x \in D$ and, of course, the inductive hypothesis. □

Remark 2.1. The condition $0 \in int(D)$ is not necessary for proving $f(0) = 0$. It is enough the condition " 0 belongs to D".

Proposition 2.1. *Let D be as before, $m \in (0,1)$, and $f : D \to \mathbb{R}$ solution of (1.3). Then, there exists a unique function $f_1 : \mathbb{R}^N \to \mathbb{R}$ solution of (1.3) in $\mathbb{R}^N$ with*

$$f_1(x) = f(x), \; x \in D.$$

Proof. Define the following recursive sequence of sets,

$$D_0 = D$$

$$D_n = c_m^n D_0, \; n \geq 1 \text{ integer}.$$

Indeed, $D_n \subseteq D_{n+1}$ for any $n \geq 0$ integer. In fact, if $x \in D_n = c_m^n D_0$ then, there is $y \in D_0$ such that $x = c_m^n y$, whence $x = c_m^{n+1} \dfrac{1}{c_m} y \in c_m^{n+1} \dfrac{1}{c_m} D_0 \subseteq c_m^{n+1} D_0 = D_{n+1}$.

On the other hand, since $0 \in int(D_0)$ by hypothesis, for any $x \in \mathbb{R}^N$, $\dfrac{1}{c_m^n} x = \left(\dfrac{m}{m+1}\right)^n x$, therefore, $\lim\limits_{n \to +\infty} \dfrac{1}{c_m^n} x = 0$, but then we may guarantee the existence of an integer $n_0 \geq 0$ such that $\dfrac{1}{c_m^n} x \in D_0$ for any $n \geq n_0$ or better, $x \in D_n$ for any $n \geq n_0$. Whence

$$\bigcup_{n=0}^{\infty} D_n = \mathbb{R}^N. \tag{2.2}$$

Define now $f_1 : \mathbb{R}^N \to \mathbb{R}$ as

$$f_1(x) = c_m^n f\Big(\frac{1}{c_m^n} x\Big) \text{ if } x \in D_n, \; n \geq 0 \text{ integer;}$$

before going any further, we must check this function is well defined. Actually, if $x \in D_n$ then $\dfrac{1}{c_m^n} x \in D_0$ and $f\Big(\dfrac{1}{c_m^n} x\Big)$ makes sense, and if $x \in D_{n_1} \cap D_{n_2}$ with, say, $n_1 < n_2$ then, it is easy to verify that

$$c_m^{n_2} f\Big(\frac{1}{c_m^{n_2}} x\Big) = c_m^{n_1} f\Big(\frac{1}{c_m^{n_1}} x\Big).$$

In any case, f_1 is well defined.

Notice that in case $n = 0$, and $x \in D$; $f_1(x) = f(x)$. Now we check that function f_1 satisfies relation (1.3) in $\mathbb{R}^N$. For that purpose take $x, y \in \mathbb{R}^N$, then there is $n \geq 0$ such that $x, y \in D_n$. But,

$$\frac{1}{c_m^n}\Big(\frac{x+y}{c_m}\Big) = \frac{1}{c_m}\left[\frac{1}{c_m^n} x + \frac{1}{c_m^n} y\right] = \frac{m}{m+1} \frac{1}{c_m^n} x + m\left(1 - \frac{m}{m+1}\right) \frac{1}{c_m^n} y \in D_0,$$

whence $\dfrac{x+y}{c_m} \in D_n$. Hence

$$\begin{aligned}
f_1\Big(\frac{x+y}{c_m}\Big) &= c_m^n f\Big(\frac{1}{c_m^n}\Big(\frac{x+y}{c_m}\Big)\Big) \\
&= c_m^n f\left(\frac{\frac{1}{c_m^n} x + \frac{1}{c_m^n} y}{c_m}\right) \\
&= c_m^n \left[\frac{f\big(\frac{1}{c_m^n} x\big) + f\big(\frac{1}{c_m^n} y\big)}{c_m}\right] \\
&= \frac{1}{c_m}\left[c_m^n f\Big(\frac{1}{c_m^n} x\Big) + c_m^n f\Big(\frac{1}{c_m^n} y\Big)\right] \\
&= \frac{f_1(x) + f_1(y)}{c_m}.
\end{aligned}$$

We finish the proof by showing that there is only one function fulfilling the above properties, and that is f_1. Suppose there is another function $f_2 : \mathbb{R}^N \to \mathbb{R}$ satisfying (1.3) and for any $x \in D_0$, $f_2(x) = f(x)$, and $f_2(0) = 0$. So if x is now in $\mathbb{R}^N$ arbitrary, by the previous lemma with $D = \mathbb{R}^N$, $f_2\big(\frac{1}{c_m^n}x\big) = \frac{1}{c_m^n}f_2(x)$ for every integer $n \geq 0$. Even more, there exists an integer $n' \geq 0$ such that $\dfrac{1}{c_m^{n'}}x \in D_0$, but then

$$f_2(x) = c_m^{n'} f_2\Big(\frac{1}{c_m^{n'}}x\Big) = c_m^{n'} f\Big(\frac{x}{c_m^{n'}}\Big) = f_1(x).$$

□

Remark 2.2. The condition $0 \in int(D)$ cannot be weakened in the previous proposition if we keep the procedure employed to extend the given function. A simple counterexample in $\mathbb{R}^2$ is the tetragon D defined by the points

$$(0,0);\ (0,1);\ (1,0);\ (1/3,1/3).$$

It is straightforward to show that D is a 1/2-convex set, $0 \notin int(D)$, and finally

$$\bigcup_{n=0}^{\infty} D_n = [0,+\infty) \times [0,+\infty) \neq \mathbb{R}^2.$$

Definition 2.1. ([7]) A function $f : \mathbb{R}^N \to \mathbb{R}$ is called additive if it satisfies the equation

$$f(x+y) = f(x) + f(y), \text{ for any } x, y \in \mathbb{R}^N.$$

Lemma 2.2. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a function satisfying (1.3), then f is additive.*

Proof. Since f satisfies (1.3), then it also fulfills conditions in Lemma 2.1, whence for any $x, y \in \mathbb{R}^N$,

$$f(x+y) = c_m \frac{1}{c_m} f(x+y) = c_m f\Big(\frac{x+y}{c_m}\Big) = c_m \left[\frac{f(x)+f(y)}{c_m}\right] = f(x) + f(y)$$

and proof is complete. □

The incoming result is similar to Theorem 13.2.1 of [7] and proof runs similar to that, so we omit it.

Theorem 2.3. *Let $D \subseteq \mathbb{R}^N$ be an m-convex set with $0 \in int(D)$, $m \in (0,1)$ and $f : D \to \mathbb{R}$ solution of (1.3). Then there exists an additive function $g : \mathbb{R}^N \to \mathbb{R}$ such that $f(x) = g(x),\ x \in D$.*

Theorem 2.4. *Let $D \subseteq \mathbb{R}^N$ be an m-convex set with $0 \in int(D)$, $m \in (0,1)$ and $f : D \to \mathbb{R}$ solution of (1.3) and continuous on D. Then, the additive function g guaranteed by the previous theorem is continuous on $\mathbb{R}^N$.*

Proof. Let x_0 be any point in $\mathbb{R}^N$. It is evident that $g|_D = f$ is continuous at x_0 when $x_0 \in D$. Otherwise, for any x arbitrary in $\mathbb{R}^N$ we know by equation (2.2) that there exists $n_0 \geq 0$ such that

$$\frac{1}{c_m^n}(x - x_0) \in D$$

for every $n \geq n_0$.

On the other hand, since f is continuous at $0 \in D$, given $\epsilon > 0$ there is a $\delta_1 = \delta_1(\epsilon) > 0$ such that $z \in D$ and $\|z - 0\| < \delta_1$ imply that

$$|f(z) - f(0)| = |f(z)| < \frac{\epsilon}{c_m^{n_0}}.$$

Set $\delta = \delta_1 c_m^{n_0} > 0$. Observe that, by taking $\|x - x_0\| < \delta$ it follows that

$$|g(x) - g(x_0)| = |g(x - x_0)| = \left| c_m^{n_0} f\Big(\frac{1}{c_m^{n_0}}(x - x_0)\Big)\right| < \epsilon.$$

So, g is also continuous in this case, and the proof is complete. $\square$

To conclude this section we cite a very well known result.

Theorem 2.5. *([7].Theorem 5.5.2., p. 139) If $f : \mathbb{R}^N \to \mathbb{R}$ is a continuous additive function, then there exists a $c \in \mathbb{R}^N$ such that*

$$f(x) = cx,$$

where $cx = \sum_{i=1}^{N} c_i x_i$ $(c = (c_1, \dots, c_N), x = (x_1, \dots, x_N))$ denotes the scalar product.

Finally, we immediately get the following important result in connection with equation (1.3).

Theorem 2.6. *Let D be as before, $m \in (0,1)$, and $f : D \to \mathbb{R}$ a continuous solution of (1.3). Then, the function g guaranteed by Theorem 2.3 is necessarily a linear function.*

3 On Stability

In this section we pursue to apply, with some modifications or adaptations, the techniques employed by Z. Kominek [14] in his work related to stability of the classical Jensen functional equation to the Jensen type equation defined by formula (1.3). In particular, we recall the definition of an ϵ-additive function. Besides that, we introduce the definition of an ϵ-m-Jensen function in the spirit of an ϵ-Jensen function defined in [14](Inequality (2), p. 499).

We close this section and the whole article with a couple of Theorems, by following ideas from [15], which refers to the Hyers-Ulam-Rassias stability of the functional inequality (3.11).

Definition 3.1. ([14]) We say that a function $f : D \subseteq \mathbb{R}^N \to \mathbb{R}$ is ϵ-additive ($\epsilon \geq 0$ is fixed) in D iff

$$|f(x+y) - f(x) - f(y)| \leq \epsilon$$

for all $x, y \in D$ such that $x + y \in D$.

Definition 3.2. ([14]) We say that a function $f : D \subseteq \mathbb{R}^N \to \mathbb{R}$ is ϵ-m-Jensen ($\epsilon \geq 0$ and $m \in (0,1]$ fixed) in D iff

$$\left| c_m f\left(\frac{x+y}{c_m}\right) - f(x) - f(y)\right| \leq \epsilon$$

for all $x, y \in D$ such that $\dfrac{x+y}{c_m} \in D$.

Clearly, the last condition is guaranteed by adding the hypothesis on D of being an m-convex set.

The next two results are going to be relevant for the demonstration of next proposition.

Lemma 3.1. *If $f : \mathbb{R}^N \to \mathbb{R}$ is an additive function and $m \in \mathbb{Q} \cap (0,1)$, then*

$$f(x) = c_m^n f\left(\frac{x}{c_m^n}\right) \tag{3.1}$$

for every $x \in \mathbb{R}^N$ and for every positive integer n.

Proof. Assume that $m = \frac{p}{q}$ (p, q positive integers and $p < q$). Then, $c_m = \frac{p+q}{p}$ and taking into account that f is additive follows that

$$f(x) = \frac{1}{p}pf(x) = \frac{1}{p}f(px) = \frac{1}{p}f\left((p+q)\frac{x}{c_m}\right) = \frac{1}{p}(p+q)f\left(\frac{x}{c_m}\right) = c_m f\left(\frac{x}{c_m}\right)$$

for every $x \in \mathbb{R}^N$. Thus, formula (3.1) is true for $n = 1$. By induction on n the general formula is easily proved. $\square$

Lemma 3.2. *([14]. Lemma 2, p. 501) If $f : (-a, a)^N \to \mathbb{R}$ ($a > 0$) is an ϵ-additive function in $(-a, a)^N$ then there exists an additive function $F : \mathbb{R}^N \to \mathbb{R}$ such that*

$$|F(x) - f(x)| \leq (5N - 1)\epsilon$$

for any $x \in (-a, a)^N$.

Proposition 3.1. *If $g : (-a, a)^N \to \mathbb{R}$ is an ϵ-m-Jensen function ($m \in \mathbb{Q} \cap (0, 1)$) then, there exist a positive constant K and a function $G : \mathbb{R}^N \to \mathbb{R}$ satisfying the equality (1.3) such that*

$$|G(x) - g(x)| \leq K\epsilon$$

for every $x \in (-a, a)^N$.

Proof. First of all, define the function $f_1 : (-a, a)^N \to \mathbb{R}$ by $f_1(x) = g(x) - g(0)$. Observe that

$$\begin{aligned}\left|c_m f_1\left(\frac{x+y}{c_m}\right) - f_1(x) - f_1(y)\right| &= \left|\left[c_m g\left(\frac{x+y}{c_m}\right) - g(x) - g(y)\right] - [c_m - 2]g(0)\right| \\ &\leq \left|c_m g\left(\frac{x+y}{c_m}\right) - g(x) - g(y)\right| + [c_m - 2]|g(0)| \\ &\leq 2\epsilon\end{aligned}$$

for every $x, y \in (-a, a)^N$. Thus, f_1 is a 2ϵ-m-Jensen function. Moreover, f_1 also satisfies

$$\begin{aligned}\left|c_m^k f_1\left(\frac{y}{c_m^k}\right) - f_1(y)\right| &= \left|c_m^k f_1\left(\frac{y}{c_m^k}\right) - \sum_{j=1}^{k-1} c_m^j f_1\left(\frac{y}{c_m^j}\right) + \sum_{j=1}^{k-1} c_m^j f_1\left(\frac{y}{c_m^j}\right) - f_1(y)\right| \\ &= \left|\sum_{j=1}^{k} c_m^{j-1}\left[c_m f_1\left(\frac{0 + \frac{y}{c_m^{j-1}}}{c_m}\right) - f_1(0) - f_1\left(\frac{y}{c_m^{j-1}}\right)\right]\right| \\ &\leq \sum_{j=1}^{k} c_m^{j-1}\left|c_m f_1\left(\frac{0 + \frac{y}{c_m^{j-1}}}{c_m}\right) - f_1(0) - f_1\left(\frac{y}{c_m^{j-1}}\right)\right| \\ &\leq \sum_{j=1}^{k} c_m^{j-1} 2\epsilon \\ &\leq (c_m^k - 1)2\epsilon \qquad (3.2)\end{aligned}$$

for every positive integer k and for every $y \in (-a, a)^N$.

Now define the set A_n for any positive integer n as follows

$$A_n := \left(-\frac{a}{c_m^{n-1}}, \frac{a}{c_m^{n-1}}\right)^N \Big\backslash \left(-\frac{a}{c_m^n}, \frac{a}{c_m^n}\right)^N.$$

Two easy-to-show properties of the collection $\{A_n\}_{n=1}^{\infty}$ are: $A_n \cap A_m = \emptyset$ iff $n \neq m$ and $\bigcup_{n=1}^{\infty} A_n \cup \{0\} = (-a,a)^N$. In other words, the collection $\{A_n\}_{n=1}^{\infty} \cup \{0\}$ is a partition of $(-a,a)^N$. Based on this condition, and since $f_1(0) = 0$, the function $f : (-a,a)^N \to \mathbb{R}$ given by the formula

$$f(x) = \frac{1}{c_m^{n-1}} f_1(c_m^{n-1}x),\ x \in A_n \cup \{0\}, n \text{ an interger}$$

is well-defined.

Now, given any $x \in (-a,a)^N$, say, $x \in A_n$, set $y = c_m^{n-1}x$. By inequality (3.2) for $k = n-1$.

$$\begin{aligned}
|f_1(x) - f(x)| &= \left| f_1(x) - \frac{1}{c_m^{n-1}} f_1(c_m^{n-1}x) \right| \\
&= \frac{1}{c_m^{n-1}} \left| c_m^{n-1} f_1\left(\frac{y}{c_m^{n-1}}\right) - f_1(y) \right| \\
&\leq \frac{1}{c_m^{n-1}} (c_m^{n-1} - 1) 2\epsilon \\
&= (1 - c_m^{1-n}) 2\epsilon \\
&\leq 2\epsilon.
\end{aligned} \tag{3.3}$$

If $x = 0$, it is trivial since $f(0) = f_1(0) = 0$.

Additionally, given $x \in (-a,a)^N$, say, $x \in A_n$, the point $x/c_m \in A_{n+1}$. Thus, we have

$$c_m f\left(\frac{x}{c_m}\right) = c_m \frac{1}{c_m^n} f_1\left(c_m^n \frac{x}{c_m}\right) = \frac{1}{c_m^{n-1}} f_1(c_m^{n-1}x) = f(x). \tag{3.4}$$

Given $x, y \in (-a,a)^N$ such that $x + y \in (-a,a)^N$. It follows, from f_1 a 2ϵ-m-Jensen function, (3.3), and (3.4) that

$$\begin{aligned}
|f(x+y) - f(x) - f(y)| &= \left| c_m f\left(\frac{x+y}{c_m}\right) - f(x) - f(y) \right| \\
&\leq c_m \left| f\left(\frac{x+y}{c_m}\right) - f_1\left(\frac{x+y}{c_m}\right) \right| + |f_1(x) - f(x)| \\
&\quad + |f_1(y) - f(y)| + \left| c_m f_1\left(\frac{x+y}{c_m}\right) - f_1(x) - f_1(y) \right| \\
&\leq 2c_m\epsilon + 6\epsilon \\
&= (2c_m + 6)\epsilon,
\end{aligned} \tag{3.5}$$

which means that f is $(2c_m + 6)\epsilon$-additive in $(-a,a)^N$ and by Lemma 3.2 there exists an additive function $G : \mathbb{R}^N \to \mathbb{R}$ such that $|G(x) - f(x)| \leq (5N-1)(2c_m + 6)\epsilon$ for every $x \in (-a,a)^N$. Then, by Lemma 3.1 G satisfies equation (1.3) and

$$\begin{aligned}
|G(x) - g(x)| &= |G(x) - g(x) + g(0) - g(0) + f(x) - f(x)| \\
&\leq |G(x) - f(x)| + |f(x) - f_1(x)| + |g(0)| \\
&\leq (5N-1)(2c_m + 6)\epsilon + 2\epsilon + \frac{\epsilon}{c_m - 2} \\
&= [(5N-1)(8 + 2/m) + (m-2)/(m-1)]\epsilon \\
&= K\epsilon
\end{aligned}$$

for every $x \in (-a,a)^N$ where $K = (5N-1)(8 + 2/m) + (m-2)/(m-1) > 0$. □

Theorem 3.3. *Let $D \subseteq \mathbb{R}^N$ be a bounded m-convex set with $0 \in int(D)$, $m \in \mathbb{Q} \cap (0,1)$, $a > 0$, p a positive integer for which*

$$i) \quad (-a,a)^N \subseteq D \qquad ii) \quad \frac{1}{c_m^p} D \subseteq (-a,a)^N, \tag{3.6}$$

and $g : D \to \mathbb{R}$ an ϵ-m-Jensen function. Then, there exist a positive constant K and a function $G : \mathbb{R}^N \to \mathbb{R}$ satisfying equality (1.3) such that

$$|G(x) - g(x)| \leq K\epsilon$$

for every $x \in D$.

Proof. Define the function $f_1 : D \to \mathbb{R}$ by $f_1(x) := g(x) - g(0)$ for every $x \in D$. Then, f_1 is a 2ϵ-m-Jensen function in D and satisfies the following inequality

$$\left| c_m^p f_1\left(\frac{y}{c_m^p}\right) - f_1(y) \right| \leq (c_m^p - 1)2\epsilon \tag{3.7}$$

for every $y \in D$ (See Proposition 3.1 for details).

Now, define $f : (-a,a)^N \to \mathbb{R}$ as in Proposition 3.1. Then, f satisfies the inequality

$$|f_1(x) - f(x)| \leq 2\epsilon \tag{3.8}$$

for every $x \in (-a,a)^N$.

Let $G : \mathbb{R}^N \to \mathbb{R}$ be an additive function such that

$$|G(x) - f(x)| \leq (5N-1)(2c_m + 6)\epsilon \tag{3.9}$$

for every $x \in (-a,a)^N$.

Taking any $x \in D$, by (3.1), part *ii)* of (3.6), (3.7), (3.8), and (3.9), we get

$$\begin{aligned} |G(x) - f_1(x)| &= \left| c_m^p G\left(\frac{x}{c_m^p}\right) - f_1(x) \right| \\ &\leq \left| c_m^p G\left(\frac{x}{c_m^p}\right) - c_m^p f_1\left(\frac{x}{c_m^p}\right) \right| + \left| c_m^p f_1\left(\frac{x}{c_m^p}\right) - f_1(x) \right| \\ &\leq c_m^p \left[\left| G\left(\frac{x}{c_m^p}\right) - f\left(\frac{x}{c_m^p}\right) \right| + \left| f\left(\frac{x}{c_m^p}\right) - f_1\left(\frac{x}{c_m^p}\right) \right| \right] + (c_m^p - 1)2\epsilon \\ &\leq c_m^p[(5N-1)(2c_m+6)+2]\epsilon + 2(c_m^p - 1)\epsilon \\ &= K_1\epsilon, \end{aligned} \tag{3.10}$$

for $K_1 = c_m^p[(5N-1)(2c_m+6)+4]$, G satisfies (1.3) by Lemma 3.1, and finally

$$|G(x) - g(x)| \leq |G(x) - f_1(x)| + |g(0)| \leq K_1\epsilon + \frac{1}{c_m - 2}\epsilon = K\epsilon$$

for every $x \in D$ where $K = K_1 + 1/(c_m - 2) > 0$. □

The next result follows ideas from [15] and refers to the Hyers-Ulam-Rassias stability of the functional inequality

$$\left| c_m f\left(\frac{x+y}{c_m}\right) - f(x) - f(y) \right| \leq \delta + \theta(\|x\|^p + \|y\|^p) \tag{3.11}$$

for $p > 0$ $(p \neq 1)$ and fixed $\delta, \theta \geq 0$.

Theorem 3.4. *Let $p > 0$ be given with $p \neq 1$. Suppose $f : \mathbb{R}^N \to \mathbb{R}$ satisfies (3.11) for all $x, y \in \mathbb{R}^N$. Then for all $x \in \mathbb{R}^N$ there exists a unique function $F : \mathbb{R}^N \to \mathbb{R}$ that satisfies (1.3) such that*

1. *if $p < 1$,*
$$|f(x) - F(x)| \leq m(\delta + |f(0)|) + \frac{\theta}{c_m^{1-p} - 1}\|x\|^p; \tag{3.12}$$
2. *if $p > 1$ and $f(0) = \delta = 0$,*
$$|f(x) - F(x)| \leq \frac{c_m^{p-1}\theta}{c_m^{p-1} - 1}\|x\|^p. \tag{3.13}$$

Proof. (1) If $p < 1$, we will show, by induction on n, the inequality
$$|c_m^{-n} f(c_m^n x) - f(x)| \leq (\delta + |f(0)|)\sum_{k=1}^{n} c_m^{-k} + \theta\|x\|^p \sum_{k=1}^{n} c_m^{(p-1)k}. \tag{3.14}$$
In fact, by choosing $y = 0$ and taking $c_m x$ instead of x in (3.11), we obtain
$$|c_m f(x) - f(c_m x)| \leq \delta + |f(0)| + \theta\|x\|^p c_m^p.$$
If we divide by c_m, it follows that (3.14) is true for $n = 1$. Now, assume that (3.14) holds for n (the inductive hypothesis). It is clear that
$$\left|c_m^{-(n+1)} f(c_m^{n+1}x) - f(x)\right| \leq c_m^{-n}\left|c_m^{-1} f(c_m^{n+1}x) - f(c_m^n x)\right| + |c_m^{-n} f(c_m^n x) - f(x)|, \tag{3.15}$$
and since (3.14) is valid for $n = 1$ and for all $x \in \mathbb{R}^N$ (in particular for $c_m^n x$ instead of x),
$$c_m^{-n}\left|c_m^{-1} f(c_m^{n+1}x) - f(c_m^n x)\right| \leq \left(\delta + |f(0)| + \theta c_m^{p(n+1)}\|x\|^p\right) c_m^{-(n+1)}. \tag{3.16}$$
By considering (3.16) and the inductive hypothesis, we get from (3.15)
$$\begin{aligned}\left|c_m^{-(n+1)} f(c_m^{n+1}x) - f(x)\right| &\leq \left(\delta+|f(0)|+\theta c_m^{p(n+1)}\|x\|^p\right)c_m^{-(n+1)} + (\delta+|f(0)|)\sum_{k=1}^{n} c_m^{-k} \\ &\quad + \theta\|x\|^p \sum_{k=1}^{n} c_m^{(p-1)k} \\ &= (\delta + |f(0)|)\sum_{k=1}^{n+1} c_m^{-k} + \theta\|x\|^p \sum_{k=1}^{n+1} c_m^{(p-1)k}.\end{aligned}$$
This proves the validity of (3.14). Now, let $x \in \mathbb{R}^N$ be given and consider the real sequence $\{c_m^{-n} f(c_m^n x)\}$. This is a Cauchy's sequence since for $n > r$ and by (3.14) (with $n - r$ and $c_m^r x$ instead of n and x, respectively), we have
$$\begin{aligned}\left|c_m^{-n} f(c_m^n x) - c_m^{-r} f(c_m^r x)\right| &= c_m^{-r}\left|c_m^{-(n-r)} f(c_m^{n-r} c_m^r x) - f(c_m^r x)\right| \\ &\leq c_m^{-r}\left[(\delta + |f(0)|)\sum_{k=1}^{n-r} c_m^{-k} + \theta c_m^{rp}\|x\|^p \sum_{k=1}^{n-r} c_m^{(p-1)k}\right] \\ &\leq c_m^{-r}\left[m(\delta + |f(0)|) + \frac{c_m^{rp}}{c_m^{1-p} - 1}\theta\|x\|^p\right] \\ &\to 0 \text{ as } r \to +\infty.\end{aligned}$$
Therefore, this sequence converges. Let us see that the function defined by
$$F(x) = \lim_{n\to+\infty} c_m^{-n} f(c_m^n x)$$

for every $x \in \mathbb{R}^N$ satisfies the required conclusion.

If $x, y \in \mathbb{R}^N$, then by (3.11)

$$\begin{aligned}\left|c_m F\left(\frac{x+y}{c_m}\right) - F(x) - F(y)\right| &= \lim_{n\to+\infty} c_m^{-n}\left|c_m f\left(\frac{c_m^n x + c_m^n y}{c_m}\right) - f(c_m^n x) - f(c_m^n y)\right| \\ &\leq \lim_{n\to+\infty} c_m^{-n}\left[\delta + \theta c_m^{np}\left(\|x\|^p + \|y\|^p\right)\right] \\ &= 0.\end{aligned}$$

Thus, F satisfies (1.3), and (3.12) follows by finding the limit in (3.14). Moreover, if $G : \mathbb{R}^N \to \mathbb{R}$ is another function that satisfies (1.3) and (3.12), then by Lemma 2.1,

$$\begin{aligned}|F(x) - G(x)| &= \left|\frac{1}{c_m^n} F(c_m^n x) - \frac{1}{c_m^n} G(c_m^n x)\right| \\ &\leq c_m^{-n}|F(c_m^n x) - f(c_m^n x)| + |f(c_m^n x) - G(c_m^n x)| \\ &\leq c_m^{-n}\left[2m(\delta + |f(0)|) + \frac{2c_m^{np}}{c_m^{1-p} - 1}\theta\|x\|^p\right] \\ &\to 0 \text{ as } n \to +\infty.\end{aligned}$$

Hence, $F(x) = G(x)$ for all $x \in \mathbb{R}^N$. So, F is unique.

(2) If $p > 1$ and $f(0) = \delta = 0$, then the proof to obtain (3.13) comes out of a similar manner, it means, by showing the inequality

$$|c_m^n f(c_m^{-n} x) - f(x)| \leq \theta\|x\|^p \sum_{k=0}^{n-1} c_m^{(1-p)k}$$

instead of (3.14) and the equality $F(x) = \lim_{n\to+\infty} c_m^n f(c_m^{-n} x)$ for all $x \in \mathbb{R}^N$. □

4 Conclusion

We have presented analysis of stability of the so called m-Jensen functional equation as a generalization of the known Jensen equation which is set for mid-convex functions. At the same time these results open the door of future research in areas like strongly m-Jensen convexity.

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Competing Interests

Authors have declared that no competing interests exist.

References

[1] Aczél J. Vorlesungen über funcktionalgleichungen und ihre anwendungen. Bassel. Stuggar; 1961.

[2] Aczél J. Einn Blick auf Funcktionalgleichungen und ihre Anwendungen. Berlin; 1962.

[3] Aczél J, Gołąb S. Funcktionalgleichungen dern Theorie der Geometrischen Objekte Warsawa; 1960.

[4] Aczél J, Gołąb S, Kuczma M, Siwek E. Das doppelverhältnis als lösung einer Funcktionalgleichungen. Annal. Polon. Math. 1960;9:183-187.

[5] Aczél J, Kiesewetter H. "Uber die reduktion der stufe bei einer klasse von Funcktionalgleichungen. Publ. Math. Debrecen. 1958;5:348-363.

[6] Kuczma M. A survey of the theory of functional equations. Publications de la Faculté d´Electronique de la l´Université ã Belgrade. 1964;130.

[7] Kuczma M. An Introduction to the Theory of Functional Equations and Inequalities. Cauchy Equations and Jensen´s Inequality. Second Edition. Birkhauser. Basel. Edited by Attila Gilányi; 2008.

[8] Hiriart-Urruty J, Lemaréchal C. Fundamentals of Convex Analysis. Springer-Verlag, Berlin-Heidelberg; 2001.

[9] Roberts A, Varberg D. Convex Functions. Academic Press. New York; 1973.

[10] Aczél J. Lectures on functional equations and their application. Mathematics in Science and Engineering, Academic Press. New York- London. 1966;19.

[11] Toader G. Some generalizations of the convexity. Proc. Colloq. Approx. Optim. Cluj-Napoca. Romania. 1984;329-338.

[12] Lara T, Quintero R, Rosales E, Sánhez J. On a generalization of the class of Jensen convex functions. Aequationes Mathematicae. 2016;90(3):569-580.

[13] Toader G. The hierarchy of convexity and some classic inequalities. J. Math. Inequ. 2009;3(3):305-313.

[14] Kominek Z. On a local stability of the Jensen functional equation. Demonstratio Mathematica. 1989;22:499-507.

[15] Jung S-M. Hyers-Ulam-Rassias stability of Jensen's equation and its application. Proceedings of the American Mathematical Society. 1998;126(11):3137-3143.

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