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On Caristi-Type Fixed Point Theorem in Cone Metric Space and Application

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Authors' contributions

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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Abstract

The aim of this paper is to generalize Caristi's fixed point theorem. For that we extend the Száz maximum principle in normed space and, we introduce a new class of functions which generalize the notion of dominated function in Caristi's fixed point theorem. As a consequence, we obtain some common fixed point results.

Keywords: Cone metric space; Caristi's fixed point theorem; lower semi continuity; strongly minihedral cone; continuous cone.

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1 Introduction

In [1, 2], Caristi proved the following result:

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying for each x in X*

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx),$$

where $\varphi : X \to [0, \infty)$ is lower semi continuous, then T has a fixed point.

The above mentioned theorem can be considered as the most important generalization of Banach principle [3] obtained in metric fixed point theory. This result is equivalent to the well known Ekeland variational principle [4, 5, 6] which is useful tool to solve problems in optimization, optimal control theory, game theory, nonlinear equations and dynamical systems [7, 8, 5, 6, 9].

Since the discovery of Caristi's fixed point theorem, there have appeared many extensions and equivalence formulations, see for instance [10, 11, 12, 13].

In 2007, [14] introduced the notion of cone metric spaces as a generalization of metric spaces replacing the set of real numbers by an ordered Banach space. The originality of their work is the introduction of the concept of convergence in cone metric spaces via the interior of the cone what they called a solid cone and then obtained some fixed point theorems for contractive mappings.

The cone metric spaces and cone normed spaces have several profound applications in fixed point theory and the numerical analysis. Some applications of cone metric spaces can be seen in [15, 16, 17, 18]. One can cite the famous Russian mathematician Kantorovich in [19] that has showed the importance of cone normed spaces in numerical analysis.

In [20, 21], Cho and Bae gave an extension of Caristi's theorem in the setting of complete cone metric spaces and, they proved that these result and Ekeland variational principle are equivalent.In the last decade, several authors have studied fixed point theorems and cone metric spaces over solid topological vector spaces (see [22, 23, 24, 25, 26, 27] and the references therein).

Brézis and Browder in [28] have provided an important general principle concerning order relations which unifies several results in nonlinear functional analysis including the Bishop-Phelps theorem [29] and Ekeland's theorem [4] as well. However in [30] it is shown that Caristi's theorem can be derived directly from the order principle of Brézis and Browder without recourse to Ekeland's Theorem.

The Brézis–Browder principle has an extensive application in mathematics theory, particularly in control theory, geometric theory and global analysis of Banach space. Its generalizations are the main concern of most researchers in the last few decades (see [31, 32, 33] and references therein). Concerning these generalizations, one can cite Turinici [33] who gave some better formulations and metric generalizations of the Altman's maximum principle [31]. In [34], Száz also proved some generalizations of Altman's results. However, these generalizations contain several superfluous hypotheses. The results of [34] were substantially improved in the subsequent paper [35]. However, this improvement still unsatisfactory from several points of view. Afterwards, Száz in [36] improve the results of [35]. Then, by using these improvements, he proved some generalizations of the most important particular cases of Brøndsted's maximality theorem [37] and gave an improved maximum principle of the Brézis–Browder ordering principle.

In this paper, we give an extension of Száz principle [36] in a normed vector space and from this preorder principle we obtained a very general Caristi's fixed point theorem in the setting of cone metric spaces, which implies most of the known Caristi-Type fixed point results and their improvements. We derive some common fixed point theorems for two single-valued mappings.

2 Preliminaries

In this section we recall some definitions and properties of cone metric spaces.

Let E be a topological vector space, a subset $P \subset E$ is called a convex cone if:

1. $P + P \subset P$,
2. for every $\lambda > 0$, $\lambda P \subset P$,
3. $P \cap (-P) = \{\theta\}$, where θ denotes the zero of E.

It is well-known that a convex cone $P \subset E$ generates a partial-ordering on E (i.e. a reflexive, antisymmetric and transitive relation) by

$$x \preceq y \Leftrightarrow y - x \in P.$$

We write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$. For $x, y \in E$, $x \ll y$ stand for $y - x \in \operatorname{int}(P)$, where $\operatorname{int}(P)$ is the interior of P.

Over this paper, $(E, \|.\|)$ is normed vector space and P is a convex cone in E.

Asadi et al. in [38] gave some examples of cones in normed space that do not enjoy the ordinary properties unlike the real line, for that we state the following definitions that will be needed in the sequel.

Definition 2.1 ([39, 40]). A cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $(x_n)_n$ is a sequence such that for some $x \in E$

$$x_1 \preceq x_2 \preceq \cdots \preceq x,$$

then there exists $\overline{x} \in E$ such that $\lim_n x_n = \overline{x}$. Equivalently, a cone P is regular if and only if every decreasing sequence which is bounded from below is convergent.

Definition 2.2 ([20, 39]). A cone P is strongly minihedral if every upper bounded non-empty subset A of E, $\sup A$ exists in E. Dually, A cone P is strongly minihedral if every lower bounded nonempty subset A of E, $\inf A$ exists in E.

Definition 2.3 ([20]). A strongly minihedral cone P is continuous if, for any bounded chain $(x_\alpha)_{\alpha\in\Gamma}$ we have

$$\inf_{\alpha\in\Gamma} \|x_\alpha - \inf\{x_\alpha;\, \alpha \in \Gamma\}\| = 0$$

and

$$\sup_{\alpha\in\Gamma} \|x_\alpha - \sup\{x_\alpha;\, \alpha \in \Gamma\}\| = 0.$$

Definition 2.4 ([14]). Let X be a nonempty set and $d : X \times X \to E$ be a mapping satisfying for all $x, y, z \in X$:

1. $\theta \preceq d(x, y)$ and $d(x, y) = \theta$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, z) \preceq d(x, y) + d(y, z)$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

The next definition is crucial throughout our work.

Definition 2.5 ([14]).

1. A sequence $(x_n)_n$ of a cone metric space (X, d) converges to a point $x \in X$ if for any $c \in \operatorname{int}(P)$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n, x) \ll c$. Denoted by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x$.
2. A sequence $(x_n)_n$ of a cone metric space (X, d) is Cauchy if for any $c \in \operatorname{int}(P)$, there exists $N \in \mathbb{N}$ such that for all $n, m > N$, $d(x_n, x_m) \ll c$.
3. A cone metric space (X, d) is called complete if every Cauchy sequence is convergent.

We have the following result.

Lemma 2.1 ([14]). *Let (X, d) be a cone metric space over a cone P in E. Then one has the following:*

1. *$Int(P) + Int(P) \subset Int(P)$ and $\lambda Int(P) \subset Int(P)$, $\lambda > 0$.*
2. *If $c \gg \theta$, then there exists $\delta > 0$ such that $\|b\| < \delta$ implies $b \ll c$.*
3. *For any given $c \gg \theta$ and $c_0 \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $\dfrac{c_0}{n_0} \ll c$.*
4. *If (a_n) , (b_n) are sequences in E such that $a_n \to a$, $b_n \to b$ and $a_n \preceq b_n$ for all $n \geq 0$, then $a \preceq b$.*

Definition 2.6 ([20]). A function $\varphi : X \to E$ is called lower semi-continuous if, for every sequence $(x_n)_n \subset X$ converging to some point $x \in X$, we have

$$\varphi(x) \preceq \liminf \varphi(x_n),$$

where $\liminf \varphi(x_n) := \sup\limits_{n\in\mathbb{N}} \inf\limits_{m\geq n} \varphi(x_m)$

3 Caristi-type Fixed Point Result

In this section we establish a full statement of the Caristi's fixed point theorem in cone metric space over a strongly minihedral cone. The main result of this section generalizes, extends and completes some results of Cho and Bae [20, 21], Caristi [1, 2], Park [41] and others.

In a nonempty set X, we define a reflexive and transitive relation $\preccurlyeq$ called a preorder and we said that $(X, \preccurlyeq)$ is preorder set. $x \in X$ is maximal element if

$$x \preccurlyeq y \Rightarrow x = y,$$

for all $y \in X$. We recall that $S(x) = \{y \in X : x \preccurlyeq y\}$.

We start with an extension of Száz principle to a normed space.

Theorem 3.1. *Let $P \subset E$ be a strongly minihedral and continuous cone, $(X, \preccurlyeq)$ be a preordered set and $a \in X$. Let $\Phi : X \times X \to E$ and suppose that the function $\gamma_\Phi : X \to E$ defined by*

$$\gamma_\Phi(x) = \sup_{x\preccurlyeq y} \Phi(x, y),$$

satisfies the following conditions:

(S1) *There exists* $m \in E$ *such that* $m \prec \gamma_{\Phi(x)}$ *for each* $x \in X$.

(S2) *There exists* $M \in E$ *such that* $\gamma_{\Phi(a) \prec M}$.

(S3) *For every nondecreasing sequence* $(x_n)_{n \in \mathbb{N}} \subset X$, *with* $x_0 = a$, *there exists some* $x \in X$ *such that* $x_n \preccurlyeq x$ *for all* $n \in \mathbb{N}$, *and* $\liminf_{n \to \infty} \Phi(x_n, x_{n+1}) \preceq \alpha$ *for some* $\alpha \in E$.

(S4) $\alpha \prec \Phi(x, y)$ *for all* $x, y \in S(a)$ *with* $x < y$.

Then there exists a maximal element $\hat{x} \in X$.

Proof. Let $x_0 = a$. Since P is strongly minihedral $\gamma_\Phi(x_0)$ exists and by the conditions (S1), (S2) we get,

$$m_0 \preceq \gamma_\Phi(x_0) \preceq M,$$

for some $m_0, M \in P$. Therefore, P is continuous we obtain

$$\sup_{x \preccurlyeq y} \|\Phi(x_0, y) - \gamma_\Phi(x_0)\| = 0,$$

hence we can choose $x_1 \in X$ with $x_0 \preccurlyeq x_1$ such that

$$\|\Phi(x_0, x_1) - \gamma_\Phi(x_0)\| < \frac{1}{2}$$

and using the fact that γ_Φ is decreasing and condition (S1) we get

$$m_1 \preceq \gamma_\Phi(x_1) \preceq \gamma_\Phi(x_0),$$

where $m_1 \in P$.

Again, P is continuous, we can choose $x_2 \in X$ with $x_1 \preccurlyeq x_2$ such that

$$\|\Phi(x_1, x_2) - \gamma_\Phi(x_1)\| < \frac{1}{3}$$

and $m_2 \preceq \gamma_\Phi(x_2) \preceq \gamma_\Phi(x_1)$ where $m_2 \in P$.

Now, by induction, it is clear that there exists an increasing sequence $(x_n)_n$ in X such that

$$\|\Phi(x_n, x_{n+1}) - \gamma_\Phi(x_n)\| < \frac{1}{n+1},$$

for all $n \in \mathbb{N}^*$.

Hence,

$$\lim_n \gamma_\Phi(x_n) = \lim_n \Phi(x_n, x_{n+1}),$$

then by the condition (S3) we have

$$\lim_n \gamma_\Phi(x_n) = \lim_n \Phi(x_n, x_{n+1}) = \liminf_n \Phi(x_n, x_{n+1}) \preceq \alpha.$$

Moreover, by the condition (S3), there exists $\bar{x} \in X$ such that $x_n \preccurlyeq \bar{x}$ for all $n \in \mathbb{N}$. Thus, in particular $a = x_0 \preccurlyeq \bar{x}$. Furthermore, since γ_Φ is decreasing, it is clear that $\gamma_\Phi(\bar{x}) \preceq \gamma_\Phi(x_n)$ for all $n \in \mathbb{N}$, and thus

$$\gamma_\Phi(\bar{x}) \preceq \lim_n \gamma_\Phi(x_n) \preceq \alpha,$$

then $\gamma_\Phi(\bar{x}) \preceq \alpha$ with $a \preceq \bar{x}$ and thus $\Phi(\bar{x}, y) \preceq \alpha$ for each $y \in X$ with $\bar{x} \preccurlyeq y$.

Now, it remains only to show that $\bar{x}$ is maximal. Assume that $\bar{x}$ is not a maximal element, that is $\bar{x} \neq y$ and $\bar{x} \prec y$ then by the condition (S4) we obtain $\alpha \prec \Phi(\bar{x}, y) \preceq \alpha$, which is a contradiction. ☐

The next definition is a generalization of dominanted function in Caristi's fixed point theorem [1].

Definition 3.1. Let $x_0 \in X$. A function $\Phi : X \times X \to E$ will be called an LZ-function if the following hold:

(K1) N-super-additivity: $\Phi(x,x) = \theta$ and $\Phi(x,y) + \Phi(y,z) \preceq \Phi(x,z)$ for each $x, y, z \in X$;

(K2) $y \mapsto \Phi(x,y)$ is upper semi continuous for each $x \in X$ (i.e. if $y_n \to y$ in X then $\limsup_{n\to\infty} \Phi(x,y_n) \preceq \Phi(x,y)$ for each $x \in X$);

(K3) $\sup_{y\in X} \Phi(x_0,y) \prec M$ for some $M \in int(P)$;

(K4) $x \mapsto \Phi(x,y)$ is bounded below for each $y \in X$.

Example 3.2. *(LZ-function in cone metric space).*

Let $X = [1,2]$ endowed by the usual distance and $E = \mathbb{R}^2$ equipped with the usual norm. Let $P = \{(x,y) \in E;\ x, y \geq 0\}$ and

$$(x,y) \preceq (u,v) \Leftrightarrow x \leq u \, and \, y \leq v.$$

Define $\Phi : X \times X \to E$ by

$$\Phi(x,y) = \left(\left(\ln\left(\frac{y}{x}\right)\right)^3, \left(\ln\left(\frac{y}{x}\right)\right)^2\right),$$

for all $x, y \in X$.

Let $x, y, z \in X$, then $\Phi(x,x) = \theta$ and

$$\begin{aligned}\Phi(x,y) + \Phi(y,z) &= \left(\left(\ln\left(\tfrac{y}{x}\right)\right)^3, \left(\ln\left(\tfrac{y}{x}\right)\right)^2\right) + \left(\left(\ln\left(\tfrac{z}{y}\right)\right)^3, \left(\ln\left(\tfrac{z}{y}\right)\right)^2\right)\\ &= \left(\left(\ln\left(\tfrac{y}{x}\right)\right)^3 + \left(\ln\left(\tfrac{z}{y}\right)\right)^3, \left(\ln\left(\tfrac{y}{x}\right)\right)^2 + \left(\ln\left(\tfrac{z}{y}\right)\right)^2\right)\\ &\preceq \left(\left(\ln\left(\tfrac{y}{x}\right) + \ln\left(\tfrac{z}{y}\right)\right)^3, \left(\ln\left(\tfrac{y}{x}\right) + \ln\left(\tfrac{z}{y}\right)\right)^2\right)\\ &= \left(\left(\ln\left(\tfrac{z}{x}\right)\right)^3, \left(\ln\left(\tfrac{z}{x}\right)\right)^2\right) = \Phi(x,z),\end{aligned}$$

then the condition (K1) holds. Since $(x,y) \mapsto \ln\left(\dfrac{y}{x}\right)$ is a continuous function we get the condition (K2). It is clear that the conditions (K3) and (K4) hold since X is bounded subset.

It is worthy to note that the last example shows that $\Phi(x,y)$ has not the form of $\varphi(x) - \psi(y)$, and it is a generalized concept of dominated function in Caristi's fixed point result.

The next lemma leads to define a preorder in a cone metric space.

Lemma 3.3. *Let (X,d) be a cone metric space and $\Phi : X \times X \to E$ be an LZ-function. A binary relation defined by*

$$x \preccurlyeq y \Leftrightarrow \quad x = y \quad or \;\; d(x,y) \preceq \Phi(x,y), \tag{3.1}$$

is a preorder on X.

Theorem 3.4. *Let (X,d) be a complete cone metric space over a strongly minihedral and continuous cone P and $T : X \to 2^X$ be a set valued map. If there exists an LZ-function $\Phi : X \times X \to E$ satisfying for each $x \in X$ there exists $y \in Tx$ such that*

$$d(x,y) \preceq \Phi(x,y), \tag{3.2}$$

then T has a fixed point in X.

Proof. We define a preorder $\preccurlyeq$ on X as (3.1).
If $x \preccurlyeq y$ and $x \neq y$, then $\theta \prec d(x,y) \preceq \Phi(x,y)$, and by the condition (K1) we obtain

$$\Phi(x,y) \preceq \Phi(x,z) - \Phi(y,z),$$

for all $y \preccurlyeq z$. Since P is strongly minihedral $\sup\limits_{x \preccurlyeq z} \Phi(x,z)$ and $\sup\limits_{y \preccurlyeq z} \Phi(y,z)$ exist so

$$\gamma_{\Phi}(y) \preceq \gamma_{\Phi}(x),$$

then $x \mapsto \gamma_{\Phi}(x)$ is decreasing and by the condition (K4), $\gamma_{\Phi}(.)$ is bounded from below.
Choose $x_0 \in X$ such that the condition (K3) holds. Then by assumption, there exists $x_1 \in Tx_0$ such that

$$d(x_0, x_1) \preceq \Phi(x_0, x_1).$$

We construct inductively a sequence $(x_n)_n$ starting at x_0 and satisfies for each $n \in \mathbb{N}$

$$d(x_n, x_{n+1}) \preceq \Phi(x_n, x_{n+1}). \tag{3.3}$$

Note that if there exists $n_0 \in \mathbb{N}$ such that $x_n = x_{n_0}$ for all $n \geq n_0$ then x_{n_0} is a fixed point of T. Assume that for all $n, m \in \mathbb{N}$ we get $x_n \neq x_m$ hence $(x_n)_n$ is an increasing sequence with respect to $\preccurlyeq$.

Note that $(\gamma_{\Phi}(x_n))_n$ is bounded sequence in P and since P is strongly minihedral, $\alpha = \inf\limits_{n \in \mathbb{N}} \gamma_{\Phi}(x_n)$ exists in E. Since, P is continuous,

$$\inf_{n \in \mathbb{N}} \|\gamma_{\Phi}(x_n) - \alpha\| = 0,$$

and hence $\lim\limits_{n \to \infty} \gamma_{\Phi}(x_n) = \alpha$ and $\alpha \preceq \gamma_{\Phi}(x_n)$ for all $n \geq 0$.

By inequality (3.3) we get for $m > n$,

$$\Phi(x_n, x_m) + \Phi(x_m, z) \preceq \Phi(x_n, z),$$

which implies

$$\Phi(x_n, x_m) + \gamma_{\Phi}(x_m) \preceq \gamma_{\Phi}(x_n),$$

then

$$d(x_n, x_m) \preceq \Phi(x_n, x_m) \preceq \gamma_{\Phi}(x_n) - \gamma_{\Phi}(x_m),$$

hence we get

$$d(x_n, x_m) \preceq \gamma_{\Phi}(x_n) - \alpha.$$

Since P is strongly minihedral and continuous we get

$$\lim_{n} \gamma_{\Phi}(x_n) = \lim_{n} \Phi(x_n, x_{n+1}) = \alpha,$$

so $(x_n)_n$ is a Cauchy sequence, then the sequence $(x_n)_n$ converges to some $\bar{x} \in X$. Note that the mapping $y \mapsto \Phi(x,y)$ is upper semi continuous then for each $n, m \in \mathbb{N}$ with $m > n$,

$$d(x_n, x_m) \preceq \Phi(x_n, x_m),$$

so taking the limit with respect to m yields

$$d(x_n, \overline{x}) \preceq \Phi(x_n, \overline{x}),$$

since d is continuous. This proves that $x_n \preccurlyeq \overline{x}$ for all $n \in \mathbb{N}$. It is clear that

$$\alpha \prec \Phi(x, y),$$

for each $x < y$ in $S(x_0)$ by the definition of α.

By Theorem 3.1 $(X, \preceq)$ has a maximal element, say, x^*. Since the condition (3.2) implies there exists $y^* \in Tx^*$ such that $x^* \preceq y^*$ it must be the case that $x^* = y^*$. ☐

If $\Phi(x, y) = \varphi(x) - \varphi(y)$ we get the following result.

Corollary 3.5 ([20]). *Let (X, d) be a complete cone metric space such that P is strongly minihedral and continuous. And, let $T : X \to X$ be a mapping satisfying for each x in X*

$$d(x, Tx) \preceq \varphi(x) - \varphi(Tx), \tag{3.4}$$

where $\varphi : X \to P$ is lower semi continuous, then T has a fixed point.

The next result is a generalization of Ekeland's variational principle in the setting of cone metric spaces.

Theorem 3.6. *Let (X, d) be a complete cone metric space over a strongly minihedral and continuous cone P and $\Phi : X \times X \to E$ be an LZ-function.*

For each $c \gg \theta$ and $x_0 \in X$ such that $\Phi(x_0, x_0) \preceq \inf_{x \in X} \Phi(x, x_0) + c$, there exists $\overline{x} \in X$ such that:
(1) $\Phi(x_0, \overline{x}) - cd(x_0, \overline{x}) \in P$;
(2) $\Phi(\overline{x}, y) - cd(\overline{x}, y) \notin P$ for each $\bar{x} \neq y$.

Proof. For each $x \in X$ we define a nonempty set $W(x)$ by

$$W(x) = \{y \in X; \quad x = y \quad \text{or} \ \ cd(x, y) \preceq \Phi(x, y)\}$$

and by continuity of $y \mapsto d(x, y)$ and lower semi continuity of $y \mapsto -\Phi(x, y)$ the set $W(x)$ is closed subset of X and hence a complete cone metric space.

Choose $x_0 \in X$ such that the condition (K3) holds. For each $x \in W(x_0)$ set

$$H(x) = \{y \in X \setminus \{x\}; \, cd(x, y) \preceq \Phi(x, y)\}$$

and define a set valued mapping

$$Tx = \begin{cases} \{x\} & \text{if, } H(x) = \emptyset \\ H(x) & \text{if, } H(x) \neq \emptyset, \end{cases}$$

then T is a self set valued mapping from $W(x_0)$ to $2^{W(x_0)}$. Indeed, if $H(x) = \emptyset$ then $Tx \in 2^{W(x_0)}$ by the definition. If $H(x) \neq \emptyset$ let $y \in H(x)$ then $y \neq x$ and $d(x, y) \preceq \Phi(x, y)$ which implies that $x \preceq y$ and since $x \in W(x_0)$ i.e. $x_0 \preccurlyeq x$ then $x_0 \preccurlyeq y$ which leads to

$$x_0 = y \quad \text{or} \ \ cd(x_0, y) \preceq \Phi(x_0, y),$$

hence $y \in W(x_0)$.

Note that for each $x \in W(x_0)$ there exists $y \in Tx$ such that

$$cd(x, y) \preceq \Phi(x, y),$$

by Theorem 3.4, T has a fixed point $\overline{x} \in W(x_0)$, it follows that $H(\overline{x}) = \emptyset$. That is, $\Phi(\overline{x}, y) \prec cd(\overline{x}, y)$ for each $y \in X \setminus \{\overline{x}\}$ and since $\overline{x} \in W(x_0)$ we get $cd(x_0, \overline{x}) \preceq \Phi(x_0, \overline{x})$. This complete the proof. ☐

4 Common Fixed Point Results

In this section, we obtain several common fixed point results for mappings satisfying more general Caristi type condition in the setting of cone metric spaces.

The next result follow easily from Theorem 3.4.

Theorem 4.1. *Let (X, d) be a complete cone metric space over a strongly minihedral and continuous cone P and $\Phi : X \times X \to E$ be an LZ-function.*
Then $(X, \preccurlyeq)$ has a maximal element, where the relation $\preccurlyeq$ is given by the formula (3.1)

Theorem 4.2. *Let (X, d) be a complete cone metric space and let P be a strongly minihedral and continuous cone of comparable elements, $\Phi : X \times X \to E$ be an LZ-function and $T, S : X \to X$ be two mappings such that for all $x \in X$,*

$$\begin{cases} d\left(x, Tx\right) \preceq \Phi\left(x, Sx\right) \\ d\left(x, Sx\right) \preceq \Phi\left(x, Tx\right). \end{cases}$$

Then there exists an element $\overline{x} \in X$ such that $T\overline{x} = S\overline{x} = \overline{x}$.

Proof. Let $x_0 \in X$ and define a subset of X as follows

$$X_0 = \left\{x \in X;\, \theta \preceq \Phi\left(x_0, x\right)\right\},$$

since $y \mapsto \Phi\left(x_0, y\right)$ is upper semi-continuous, X_0 is a nonempty closed subset of X.

We define a preorder $\preccurlyeq$ on X_0 as Lemma 3.3, that is for each $x, y \in X_0$

$$x \preccurlyeq y \Leftrightarrow x = y \text{ or } d\left(x, y\right) \preceq \Phi\left(x, y\right),$$

then using Theorem 4.1, we conclude that $(X_0, \preccurlyeq)$ has a maximal element $\overline{x} \in X_0$ such that

$$\begin{cases} d\left(\overline{x}, T\overline{x}\right) \preceq \Phi\left(\overline{x}, S\overline{x}\right) \\ d\left(\overline{x}, S\overline{x}\right) \preceq \Phi\left(\overline{x}, T\overline{x}\right). \end{cases} \tag{4.1}$$

By hypothesis, the elements of P are comparable. Then either $\Phi\left(\overline{x}, S\overline{x}\right) \preceq \Phi\left(\overline{x}, T\overline{x}\right)$ or $\Phi\left(\overline{x}, T\overline{x}\right) \preceq \Phi\left(\overline{x}, S\overline{x}\right)$. Without loss of generality, we may assume that $\Phi\left(\overline{x}, S\overline{x}\right) \preceq \Phi\left(\overline{x}, T\overline{x}\right)$ and so from the inequalities (4.1), we obtain

$$d\left(\overline{x}, T\overline{x}\right) \preceq \Phi\left(\overline{x}, T\overline{x}\right),$$

then $\overline{x} \preccurlyeq T\overline{x}$ which implies $T\overline{x} = S\overline{x} = \overline{x}$. □

If the LZ-function is written as follows $\Phi(x, y) = \varphi(x) - \psi(y)$ one can obtain the following result.

Corollary 4.3. *Let (X, d) be a complete cone metric space and let P be a strongly minihedral and continuous cone of comparable elements, $\varphi, \psi : X \to E$ be two lower semi-continuous functions and $T, S : X \to X$ be two mappings such that for all $x \in X$*

$$\begin{cases} d\left(x, Tx\right) \preceq \varphi\left(x\right) - \psi\left(Sx\right) \\ d\left(x, Sx\right) \preceq \varphi\left(x\right) - \psi\left(Tx\right). \end{cases}$$

Then there exists an element $\overline{x} \in X$ such that $T\overline{x} = S\overline{x} = \overline{x}$.

If (X, d) is a complete metric space then we have the following corollary.

Corollary 4.4. *Let (X,d) be a complete metric space, $\varphi,\psi : X \to \mathbb{R}_+$ be two lower semi-continuous functions and $T,S : X \to X$ be two mappings such that for all $x \in X$*

$$\begin{cases} d(x,Tx) \le \varphi(x) - \psi(Sx) \\ d(x,Sx) \le \varphi(x) - \psi(Tx). \end{cases}$$

Then there exists an element $\overline{x} \in X$ such that $T\overline{x} = S\overline{x} = \overline{x}$.

Theorem 4.5. *Let (X,d) be a complete cone metric space and let P a strongly minihedral and continuous of comparable elements cone and $\Phi,\Psi : X \times X \to E$ be two LZ-functions. Let $T,S : X \to X$ be two continuous mappings such that for all $x \in X$,*

$$\begin{cases} d(x,Tx) \preceq \Phi(x,Sx) \\ d(x,Sx) \preceq \Psi(x,Tx). \end{cases}$$

Assume that there exists $x_0 \in X$ such that $\Phi(x_0,Sx_0) \preceq \Phi(x_0,Tx_0)$ and $\Psi(x_0,Tx_0) \preceq \Psi(x_0,Sx_0)$. Then there exists an element $\overline{x} \in X$ such that $T\overline{x} = S\overline{x} = \overline{x}$.

Proof. Define a subset X_0 of X as follows:

$$X_0 = \{x \in X; \Psi(x,Tx) \preceq \Phi(x,Sx) \text{ and } \Phi(x,Sx) \preceq \Psi(x,Tx)\},$$

then $X_0 \neq \emptyset$ and since T,S,Ψ and Φ are upper semi continuous, X_0 is a complete subset of X.

We define a preorder $\preccurlyeq$ on X_0 as follows

$$x \preccurlyeq y \Leftrightarrow d(x,y) \preceq \frac{1}{2}\left(\Phi(x,y) + \Psi(x,y)\right),$$

for each $x,y \in X_0$. Note that the sum of two LZ-functions is an LZ-function. Hence, using Theorem 4.1, we conclude that $(X_0,\preccurlyeq)$ has a maximal element $\overline{x} \in X_0$ such that

$$\begin{cases} d(\overline{x},T\overline{x}) \preceq \Phi(\overline{x},S\overline{x}) \\ d(\overline{x},S\overline{x}) \preceq \Psi(\overline{x},T\overline{x}). \end{cases} \tag{4.2}$$

Since P has a comparable elements, either $d(\overline{x},S\overline{x}) \preceq d(\overline{x},T\overline{x})$ or $d(\overline{x},T\overline{x}) \preceq d(\overline{x},S\overline{x})$. Without loss of generality, we may assume that $d(\overline{x},T\overline{x}) \preceq d(\overline{x},S\overline{x})$ we get by the inequalities (4.2)

$$d(\overline{x},T\overline{x}) \preceq \frac{1}{2}\left(\Phi(\overline{x},S\overline{x}) + \Psi(\overline{x},T\overline{x})\right) \preceq \frac{1}{2}\left(\Phi(\overline{x},T\overline{x}) + \Psi(\overline{x},T\overline{x})\right),$$

i.e. $\overline{x} \preccurlyeq T\overline{x}$ which implies that $\overline{x} = T\overline{x}$, by the inequalities (4.2) we obtain $\overline{x} = T\overline{x} = S\overline{x}$.

The same conclusion holds in the case $d(\overline{x},S\overline{x}) \preceq d(\overline{x},T\overline{x})$. □

5 Conclusion

We shall give a brief summary of our manuscript:

a In Section (2), we recalled the most useful results and definitions in cone metric space.

b In Section (3), we started with an extension of Száz maximum principle in normed space (Theorem 3.1), afterward, we have introduced a new class of functions (Definition 3.1) which allows us to give a more generalized version of Caristi's fixed point theorem (Theorem 3.4).

c In Section (4), we have applied our results of section (3) to obtain several common fixed point theorems.

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Competing Interests

Authors have declared that no competing interests exist.

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