Asian Research Journal of Mathematics
3(3): 1-8, 2017; Article no.ARJOM.32312
ISSN: 2456-477X

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Square Fibonacci Numbers and Square Lucas Numbers

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/ARJOM/2017/32312
Editor(s):
(1) Danilo Costarelli, Department of Mathematics and Computer Science, University of Perugia, Italy.
Reviewers:
(1) Can Kizilates, Bulent Ecevit University, Turkey.
(2) Arzu Ozkoc, Duzce University, Turkey.
(3) Sergio Falcon, University of Las Palmas De Gran Canaria, Spain.
Complete Peer review History: http://www.sciencedomain.org/abstract/18559

Received: 20th February 2017
Accepted: 31st March 2017
Published: 8th April 2017

Original Research Article

Abstract

In this paper we deduce the integer n satisfying $L_n = 3x^2$ and $F_n = 3x^2$, respectively after obtaining the Legendre-Jacobi symbol $\left(\frac{-3}{L_k}\right) = -1$ and $\left(\frac{-7}{L_k}\right) = -1$ for $L_k \equiv 3 \pmod 4$ with $2|k, 3 \nmid k$.

Keywords: Lucas and Fibonacci numbers.

2010 Mathematics Subject Classification: 11B39.

1 Introduction

We may define the Fibonacci numbers, F_n, by

$$F_0 = 0, \qquad F_1 = 1, \qquad \text{and,} \qquad F_{n+2} = F_{n+1} + F_n.$$

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Fibonacci numbers associated with Lucas numbers Ln, which we may define by

$$L_0 = 2, \qquad L_1 = 1, \qquad \text{and,} \qquad L_{n+2} = L_{n+1} + L_n.$$

Fibonacci numbers and Lucas numbers can also be extended to negative index n satisfying

$$F_{-n} = (-1)^{n+1}F_n \qquad \text{and} \qquad L_{-n} = (-1)^n L_n. \tag{1.1}$$

We shall require the following results which are easily proved from the definitions. Throughout this paper, the following n, m, and k will denote integers, not necessarily positive, and r will denote a non-negative integer. Also wherever it occurs, k will denote an even integer, not divisible by 3 :

$$2F_{m+n} = F_m L_n + F_n L_m, \tag{1.2}$$

$$2L_{m+n} = 5F_m F_n + L_m L_n, \tag{1.3}$$

$$L_{2m} = L_m^2 + (-1)^{m-1}2, \tag{1.4}$$

$$\gcd(F_n, L_n) = 2 \qquad \text{if} \quad 3|n, \tag{1.5}$$

$$\gcd(F_n, L_n) = 1 \qquad \text{if} \quad 3 \nmid n, \tag{1.6}$$

$$2|L_n \qquad \text{if and only if} \quad 3|n, \tag{1.7}$$

$$3|L_n \qquad \text{if and only if} \quad n \equiv 2 \pmod 4, \tag{1.8}$$

$$L_k \equiv 3 \pmod 4 \qquad \text{if} \quad 2|k, 3 \nmid k, \tag{1.9}$$

$$L_{n+2k} \equiv -L_n \pmod{L_k}, \tag{1.10}$$

$$F_{n+2k} \equiv -F_n \pmod{L_k}. \tag{1.11}$$

In this article, we mainly obtain the following results based on J. H. E. Cohn's method in [1] and [2]. In fact, N. Robbins gave the solutions of Theorem 1.2 and Theorem 1.3 in [3] for a natural number n. But here we find those solutions for integer n by using another method:

Theorem 1.1. *Let $L_k \equiv 3 \pmod 4$ with $2|k, 3 \nmid k$ as in (1.9). Then we have*

(a)

$$\left(\frac{-3}{L_k}\right) = -1.$$

(b)

$$\left(\frac{-7}{L_k}\right) = -1.$$

Theorem 1.2. *If $L_n = 3x^2$, then $n = 2$ or $n = -2$.*

Theorem 1.3. *If $F_n = 3x^2$, then $n = 0$ or $n = 4$.*

2 Proofs of Theorem 1.1, Theorem 1.2, and Theorem 1.3

Proposition 2.1. *(See [1], [2]) We have*

(a) *If $L_n = x^2$, then $n = 1$ or 3.*
(b) *If $L_n = 2x^2$, then $n = 0$ or ± 6.*
(c) *If $F_n = x^2$, then $n = 0$, ± 1, 2, or 12.*
(d) *If $F_n = 2x^2$, then $n = 0$, ± 3, or 6.*

We can easily show the following lemma by Proposition 2.1 (a) :

Lemma 2.1. *If $L_n = 4x^2$, then $n = 3$.*

Proof. Now it yields that

$$L_n = 4x^2 = (2x)^2 = y^2$$

and so by Proposition 2.1 (a) we have $n = 1$ or 3. Then

$$L_1 = 1 \qquad \text{and} \qquad L_3 = 4$$

so the solution is $n = 3$.

□

In another point of view we try to prove Lemma 2.1 and in that process we find Theorem 1.1.

Proof of Theorem 1.1. (a) Since Lemma 2.1 has solution only $n = 3$ thus for $n \equiv 2 \pmod 8$ there does not exist a solution. Then $L_2 = 3$, whereas if $n \neq 2$ we can write $n = 2 + 2 \cdot 3^r \cdot k$ with $2|k, 3 \nmid k$ and so by (1.10),

$$4L_n = 4L_{2+2\cdot 3^r \cdot k} \equiv -4L_2 \pmod{L_k} \equiv -4 \cdot 3 \pmod{L_k}.$$

Therefore by (1.9), the following Legendre-Jacobi symbol must satisfy

$$-1 = \left(\frac{-4 \cdot 3}{L_k}\right) = \left(\frac{-3}{L_k}\right)$$

since -1 is a non-residue of L_k.

(b) In a similar manner to part (a), if $n \equiv 4 \pmod 8$ then $L_4 = 7$, whereas if $n \neq 4$ we can write $n = 4 + 2 \cdot 3^r \cdot k$ with $2|k, 3 \nmid k$ and so by (1.10),

$$4L_n = 4L_{4+2\cdot 3^r \cdot k} \equiv -4L_4 \pmod{L_k} \equiv -4 \cdot 7 \pmod{L_k}.$$

so by (1.9), we conclude that

$$-1 = \left(\frac{-4 \cdot 7}{L_k}\right) = \left(\frac{-7}{L_k}\right).$$

□

Lemma 2.2. *If $F_n = 4x^2$, then $n = 0$ or $n = 12$.*

Proof. We can see that

$$F_n = 4x^2 = (2x)^2 = y^2$$

and so by Proposition 2.1 (c) we have $n = 0, \pm 1, 2,$ or 12. Therefore

$$F_0 = 0, \quad F_1 = 1, \quad F_{-1} = 1, \quad F_2 = 1, \quad \text{and} \quad F_3 = 144$$

so the solution is $n = 0$ or $n = 12$.

□

Another Proof of Lemma 2.2. (i) If $n \equiv 1 \pmod 4$, then $F_1 = 1$, whereas if $n \neq 1$, $n = 1 + 2 \cdot 3^r \cdot k$ with $2|k, 3 \nmid k$ and so by (1.11),

$$4F_n = 4F_{1+2\cdot 3^r \cdot k} \equiv -4F_1 \pmod{L_k} \equiv -4 \cdot 1 \pmod{L_k}. \tag{2.1}$$

Now by (1.9) we can know that

$$L_k = 4l_1 + 3 \qquad \text{for} \quad l_1 \in \mathbb{Z} \tag{2.2}$$

and so (2.1) implies that

$$\left(\frac{4F_n}{L_k}\right) = \left(\frac{-4 \cdot 1}{L_k}\right) = \left(\frac{-1}{L_k}\right) = (-1)^{\frac{4l_1+3-1}{2}} = -1$$

thus $4F_n \neq y^2$, that is, $F_n \neq 4x^2$.

(ii) If $n \equiv 7 \pmod 8 \equiv -1 \pmod 8$, then $F_{-1} = 1$, whereas if $n \neq -1$, $n = -1 + 2 \cdot 3^r \cdot k$ with $2|k, 3 \nmid k$ and so by (1.11),

$$4F_n = 4F_{-1+2\cdot 3^r \cdot k} \equiv -4F_{-1} \pmod{L_k} \equiv -4 \cdot 1 \pmod{L_k}.$$

Then from (2.2) we have

$$\left(\frac{4F_n}{L_k}\right) = \left(\frac{-4 \cdot 1}{L_k}\right) = \left(\frac{-1}{L_k}\right) = (-1)^{\frac{4l_1+3-1}{2}} = -1$$

and so $4F_n \neq y^2$, that is, $F_n \neq 4x^2$.

(iii) If $n \equiv 3 \pmod 8$, then $F_3 = 2$, whereas if $n \neq 3$, $n = 3 + 2 \cdot 3^r \cdot k$ with $4|k, 3 \nmid k$ and so by (1.11),

$$4F_n = 4F_{3+2\cdot 3^r \cdot k} \equiv -4F_3 \pmod{L_k} \equiv -4 \cdot 2 \pmod{L_k}. \tag{2.3}$$

Here since $4|k$ we can put $k = 2k_1$ for $2|k_1$ and $3 \nmid k_1$. Thus by (1.4) and (2.2) we deduce that

$$L_k = L_{2k_1} = L_{k_1}^2 + (-1)^{k_1 - 1} 2 = L_{k_1}^2 - 2 = (4l_1 + 3)^2 - 2 \equiv 7 \pmod 8$$

so we can write

$$L_k = 8l_2 + 7 \qquad \text{for} \quad l_2 \in \mathbb{Z}. \tag{2.4}$$

Therefore (2.3) shows that

$$\left(\frac{4F_n}{L_k}\right) = \left(\frac{-4\cdot 2}{L_k}\right) = \left(\frac{-1}{L_k}\right)\left(\frac{2}{L_k}\right) = (-1)^{\frac{8l_2+7-1}{2}}(-1)^{\frac{(8l_2+7)^2-1}{8}} = -1$$

and so $F_n \neq 4x^2$.

(iv) Suppose that n is even and $F_n = 4x^2$. Then by (1.2) we obtain

$$4x^2 = F_n = F_{\frac{n}{2}} L_{\frac{n}{2}}$$

and so (1.5) and (1.6) give three possibilities :

(a) $3|n$, $F_{\frac{n}{2}} = 2y^2$; $L_{\frac{n}{2}} = 2z^2$. By Proposition 2.1 (b), the second of these is satisfied only by

$$\frac{n}{2} = 0, 6, \text{ or } -6 \Leftrightarrow n = 0, 12, \text{ or } -12.$$

Also by Proposition 2.1 (d), the first of these is satisfied only by

$$\frac{n}{2} = 0, 3, -3, \text{ or } 6 \Leftrightarrow n = 0, 6, -6, \text{ or } 12.$$

Thus the common factors are $n = 0$ and 12.

(b) $3 \nmid n$, $F_{\frac{n}{2}} = y^2$; $L_{\frac{n}{2}} = 4z^2$. Lemma 2.1 shows that

$$\frac{n}{2} = 3 \Leftrightarrow n = 6,$$

which is contradiction to $3 \nmid n$.

(c) $3 \nmid n$, $F_{\frac{n}{2}} = 4y^2$; $L_{\frac{n}{2}} = z^2$. From Proposition 2.1 (a) the latter is satisfied only for

$$\frac{n}{2} = 1 \text{ or } 3 \Leftrightarrow n = 2 \text{ or } 6.$$

However the last of these must be rejected since it does not satisfy $3 \nmid n$ and also because $F_1 = 1 \neq 4y^2$ we delete $n = 2$ case.

This concludes the proof.

□

Proof of Theorem 1.2. (i) If $n \equiv 1 \pmod 4$, then $L_1 = 1$, whereas if $n \neq 1$, we can write $n = 1 + 2\cdot 3^r \cdot k$ with $2|k, 3 \nmid k$ and so by (1.10),

$$3L_n = 3L_{1+2\cdot 3^r\cdot k} \equiv -3L_1 \pmod{L_k} \equiv -3\cdot 1 \pmod{L_k}.$$

Thus by Theorem 1.1 (a) we have

$$\left(\frac{3L_n}{L_k}\right) = \left(\frac{-3}{L_k}\right) = -1$$

and so $L_n \neq 3x^2$.

(ii) If $n \equiv 3 \pmod 4$, then $L_3 = 4$, whereas if $n \neq 3$, $n = 3 + 2 \cdot 3^r \cdot k$ with $2|k, 3 \nmid k$ and so by (1.10),

$$3L_n = 3L_{3+2\cdot 3^r \cdot k} \equiv -3L_3 \pmod{L_k} \equiv -3 \cdot 4 \pmod{L_k}.$$

Then from Theorem 1.1 (a) we observe that

$$\left(\frac{3L_n}{L_k}\right) = \left(\frac{-3 \cdot 4}{L_k}\right) = \left(\frac{-3}{L_k}\right) = -1$$

and so $L_n \neq 3x^2$.

(iii) If $n \equiv 2 \pmod 4$, then $L_{\pm 2} = 3 = 3x^2$, whereas if $n \neq \pm 2$, $n = 2 + 2 \cdot 3^r \cdot k$ with $2|k, 3 \nmid k$ and so by (1.10),

$$3L_n = 3L_{2+2\cdot 3^r \cdot k} \equiv -3L_2 \pmod{L_k} \equiv -3 \cdot 3 \pmod{L_k}.$$

Thus by (2.2) we deduce that

$$\left(\frac{3L_n}{L_k}\right) = \left(\frac{-3 \cdot 3}{L_k}\right) = \left(\frac{-1}{L_k}\right) = (-1)^{\frac{4l_1+3-1}{2}} = -1$$

and so $L_n \neq 3x^2$.

(iv) If $n \equiv 0 \pmod 8$, then $L_0 = 2$, whereas if $n \neq 0$, $n = 2 \cdot 3^r \cdot k$ with $4|k, 3 \nmid k$ and so by (1.10),

$$3L_n = 3L_{2\cdot 3^r \cdot k} \equiv -3L_0 \pmod{L_k} \equiv -3 \cdot 2 \pmod{L_k}.$$

Now because of $4|k$ we can apply (2.4) to L_k. Then also by Theorem 1.1 (a) we note that

$$\left(\frac{3L_n}{L_k}\right) = \left(\frac{-3 \cdot 2}{L_k}\right) = \left(\frac{-3}{L_k}\right)\left(\frac{2}{L_k}\right) = -(-1)^{\frac{(8l_2+7)^2-1}{8}} = -1 \tag{2.5}$$

and so $L_n \neq 3x^2$.

(v) If $n \equiv 4 \pmod 8$, then $L_4 = 7$, whereas if $n \neq 4$, $n = 4 + 2 \cdot 3^r \cdot k$ with $2|k, 3 \nmid k$ and so by (1.10),

$$3L_n = 3L_{4+2\cdot 3^r \cdot k} \equiv -3L_4 \pmod{L_k} \equiv -3 \cdot 7 \pmod{L_k}.$$

Therefore from Theorem 1.1 we obtain

$$\left(\frac{3L_n}{L_k}\right) = \left(\frac{-3 \cdot 7}{L_k}\right) = \left(\frac{-3}{L_k}\right)\left(\frac{7}{L_k}\right) = -1 \cdot 1 = -1$$

since Theorem 1.1 (b) and (2.2) show that

$$-1 = \left(\frac{-7}{L_k}\right) = \left(\frac{-1}{L_k}\right)\left(\frac{7}{L_k}\right) = (-1)^{\frac{4l_1+3-1}{2}}\left(\frac{7}{L_k}\right) = -\left(\frac{7}{L_k}\right)$$

thus

$$\left(\frac{7}{L_k}\right) = 1.$$

□

Proof of Theorem 1.3. (i) If $n \equiv 1 \pmod 4$, then $F_1 = 1$, whereas if $n \neq 1$, we can write $n = 1 + 2 \cdot 3^r \cdot k$ with $2|k, 3 \nmid k$ and so by (1.11),

$$3F_n = 3F_{1+2\cdot 3^r \cdot k} \equiv -3F_1 \pmod{L_k} \equiv -3 \cdot 1 \pmod{L_k}.$$

Thus by Theorem 1.1 (a) we see that

$$\left(\frac{3F_n}{L_k}\right) = \left(\frac{-3}{L_k}\right) = -1.$$

(ii) If $n \equiv 3 \pmod 8$, then $F_3 = 2$, whereas if $n \neq 3$, $n = 3 + 2 \cdot 3^r \cdot k$ with $4|k, 3 \nmid k$ and so by (2.5) we conclude that $F_n \neq 3x^2$.

(iii) If $n \equiv 7 \pmod 8 \equiv -1 \pmod 8$, then $F_{-1} = 1$, whereas if $n \neq -1$, $n = -1 + 2 \cdot 3^r \cdot k$ with $2|k, 3 \nmid k$ and so by (1.11),

$$3F_n = 3F_{-1+2\cdot 3^r \cdot k} \equiv -3F_{-1} \pmod{L_k} \equiv -3 \cdot 1 \pmod{L_k}.$$

Thus Theorem 1.1 (a) implies that

$$\left(\frac{3F_n}{L_k}\right) = \left(\frac{-3}{L_k}\right) = -1.$$

(iv) Suppose that n is even and $F_n = 3x^2$. Then by (1.2) we obtain

$$3x^2 = F_n = F_{\frac{n}{2}} L_{\frac{n}{2}}$$

and so (1.5) and (1.6) give four possibilities :

(a) $3|n$, $F_{\frac{n}{2}} = 6y^2$; $L_{\frac{n}{2}} = 2z^2$. By Proposition 2.1 (b), the second of these is satisfied only by

$$\frac{n}{2} = 0, 6, \text{ or } -6 \Leftrightarrow n = 0, 12, \text{ or } -12.$$

And since

$$F_0 = 0, \qquad F_6 = 8, \qquad \text{and} \qquad F_{-6} = -8,$$

we choose $n = 0$.

(b) $3|n$, $F_{\frac{n}{2}} = 2y^2$; $L_{\frac{n}{2}} = 6z^2$. By Proposition 2.1 (d), the first of these is satisfied only by

$$\frac{n}{2} = 0, 3, -3, \text{ or } 6 \Leftrightarrow n = 0, 6, -6, \text{ or } 12.$$

Then since

$$L_0 = 2, \qquad L_3 = 4, \qquad L_{-3} = -4, \qquad \text{and} \qquad L_6 = 18,$$

there is no solution.

(c) $3 \nmid n$, $F_{\frac{n}{2}} = 3y^2$; $L_{\frac{n}{2}} = z^2$. Proposition 2.1 (a) requests

$$\frac{n}{2} = 1 \text{ or } 3 \Leftrightarrow n = 2 \text{ or } 6.$$

However $n = 6$ must be rejected since it does not satisfy $3 \nmid n$ also $n = 2$ is deleted by $F_1 = 1 \neq 3y^2$.
(d) $3 \nmid n$, $F_{\frac{n}{2}} = y^2$; $L_{\frac{n}{2}} = 3z^2$. From Theorem 1.2 we have

$$\frac{n}{2} = 2 \text{ or } -2 \Leftrightarrow n = 4 \text{ or } -4.$$

Similarly by Proposition 2.1 (c), we note that

$$\frac{n}{2} = 0, 1, -1, 2, \text{ or } 12 \Leftrightarrow n = 0, 2, -2, 4, \text{ or } 24$$

and so the common factor is $n = 4$.

Hence we have in all the two values, $n = 0$ or $n = 4$.

□

3 Conclusion

We can find more general solutions of square Fibonacci numbers and square Lucas numbers in [3], [4], and [5].

Competing Interests

Author has declared that no competing interests exist.

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