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Sheffer Polynomials and their Delta Operators

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Authors' contributions

This work was carried out in collaboration between both authors. Author AM designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript and managed literature searches. Author CE managed the analyses of the study and literature searches. Both authors read and approved the final manuscript.

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Abstract

Aims/objectives: In this paper, we study the Sheffer polynomials through the sequential representation of delta operator in Finite Operator Calculus. The major objective is to investigate the characterization of the delta operator for the Simple Laguerre, the Boole and Mittag-Leffer polynomials. From our investigation, we derive many interesting Propositions for the above polynomials.

Keywords: *Delta operator; Basic polynomial sequences; Sheffer polynomials; Leguerre polynomial; Boole polynomial; Mittag-Leffer polynomial.*

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1 Introduction

I.M. Sheffer introduced the sequence of Sheffer polynomials in 1939. It is a polynomial sequence in which the index of each polynomial equal its degree, satisfying some conditions related to the Umbral Calculus in Combinatorics. Sheffer polynomials are classified and many important properties are derived by Rainvile [1]. Steven Roman [2] studied this Sheffer polynomials and derived many wonderful results for Combinatorics Theory. In 1975, G.C.Rota[3] introduced Finite Operator Calculus which is a systematic study of operators on the algebra of polynomials. It contains a detailed study of basic polynomials and Sheffer polynomials associated with delta operator. After briefly summarizing and analysing Rota [3] critically, many new results evolved in this paper are discussed.

The investigation of the properties of Sheffer set in [4], the symbolic approach of Sheffer polynomials in [5] and many interesting identities of Sheffer set in [6] are effective study of the Sheffer polynomials.

The rest of the paper is organized as follows. In the section 2, the basic concepts in Finite Operator Calculus is given. The sequential representation of the delta operator is discussed in the section 3. The characterization of delta operator for the Laguerre, the Boole and the Mittag-Leffer polynomials are investigated in the section 4 and all the results are consolidated in a Table. Finally, we conclude our paper with some extension associated with our new approach of Sheffer polynomials.

2 Basics in Finite Operator Calculus

In this section, we recall some basic definitions and theorems of the Finite Operator Calculus, as it has been introduced by G.C.Rota. The proofs of known results are skipped, but they are easily read from the reference G.C.Rota [3].

Let F be a Field of characteristic zero, preferably the real number field. Let $p(x)$ be a polynomial in one variable defined over F. A sequence of polynomials is $\{p_n(x)/n \in \mathbb{Z}^+ \cup \{0\}\}$, where $p_n(x)$ is exactly of degree n.

We study certain special type of operators in this section.

Definition 1.1

i An operator E^a is said to be a shift operator if $E^a p(x) = p(x + a)$, for all polynomials $p(x)$ in one variable and for all real a in the field F.

ii A linear operator T which commutes with all shift operators is called a shift invariant. In symbols, $TE^a = E^a T, \quad \forall a \in F$.

iii A shift invariant operator Q satisfying that Qx is a non zero constant is called a delta operator.

Thus every delta operator Q is a shift invariant. But a shift invariant operator need not be a delta operator.

Definition 1.2
Let Q be the delta operator. A polynomial sequence $p_n(x)$ is called the sequence of basic polynomials for Q if :

i. $p_0(x) = 1$.

ii. $p_n(0) = 0, \quad whenever\ n > 0$.

iii. $Qp_n(x) = np_{(n-1)}(x)$.

The common examples for basic polynomials are listed below.
The trivial example is the Monomials $\{x^n : n \in \mathbb{Z}^+ \cup \{0\}\}$.
The sequence of " Lower-Factorials " is defined by

$$[x]_n = x(x-1)(x-2)\cdots(x-n+1).$$

The product is understood to be 1 if $n = 0$, since it is in that case an empty product. This polynomial sequence is of basic polynomials.
The " Upper-Factorials "

$$[x]^n = x(x+1)(x+2)\cdots(x+n-1).$$

are basic polynomial sequence.
The " Abel polynomials"

$$A_n(x) = x\ (x-na)^{n-1}, \quad n \in \mathbb{Z}^+ \cup \{0\}\ and\ a \neq 0$$

are of basic polynomials.

Definition 1.3

A polynomial sequence $p_n(x)$ is said to be a binomial type if it satisfies the infinite sequence of following identities

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)\ p_{n-k}(y), \quad n = 0, 1, 2, \ldots$$

The simplest sequence of Binomial type is $\{x^n\}$.

Definition 1.4

A Polynomial sequence $s_n(x)$ is called a Sheffer set or a set of Sheffer polynomials for the delta operator Q if

i $s_0(x) = c \neq 0$,

ii $Qs_n(x) = ns_{n-1}(x)$

It is also called " polynomial sequence of type zero ". Sheffer polynomials are a large class of polynomials that include Monomials $\{x^n\}$, Abel polynomials, Falling factorial and Raising factorial polynomials, Hermite polynomials, Bernoulli polynomials, Boole polynomials, Laguerre polynomials, Mittag-Leffer polynomials, Mott polynomials and many others.

The delta operators possess many of the properties of the usual derivative D. The following theorems are good examples.

Theorem 1.5

i Every delta operator has a unique sequence of basic polynomials.

ii. If Q is a delta operator, then $Qa = 0$ for every constant $'a'$.

iii. If $p(x)$ is a polynomial of degree n, then $Qp(x)$ is a polynomial of degree $n-1$.

The following result establishes the connection between delta operator and the binomial type sequences. Moreover, it gives the necessary and sufficient conditions for basic polynomial sequence for some delta operator Q.

Theorem 1.6

i. If $p_n(x)$ is a basic sequence for some delta operator Q, then it is a sequence of polynomials of Binomial type.

ii. If $p_n(x)$ is a sequence of polynomials of Binomial type, then it is a basic sequence for some delta operator.

Thus we have $p_n(x)$ is a basic polynomials sequence for some delta operator Q if and only if it is a sequence of polynomials of Binomial type.

Iterating the property (iii) in Definition (1.2) of basic polynomials we obtain

$$(Q^k\ p_n)(x) = [n]_k\ p_{n-k}.$$

Where $[n]_k = n(n-1)(n-2)\cdots(n-k+1)$.
For $k = n$, we have

$$(Q^n p_n)(0) = n!$$

For $k < n$,

$$(Q^k\ p_n)(0) = (0)\ holds.$$

Since any polynomial q is a linear combination of the basic polynomials, we have

$$q(x) = \sum_{k=0}^{deg(q)} \frac{Q^k q(0)}{k!}\ p_k(x)$$

Moreover, by choosing $q := E^y q$, we get

$$q(x+y) = \sum_{k=0}^{deg(q)} \frac{Q^k q(y)}{k!}\ p_k(x)$$

This identity is the good starting point which permits us to obtain the expansion of a shift-invariant operator in terms of a delta operator and its power. The sequential representation of delta operator in [7] is discussed in the next section. This sequential expression is useful to investigate the delta operator for many Sheffer polynomials.

3 Sequential Representation of Delta Operator

Rota suggested an open question which is given below

Work out formulae for $p_n(Q)$, when $p_n(x)$ is a Sheffer set relative to the delta operator Q.

The above open question is slightly modified as

Work out formulae for $Q(p_n)$, when $p_n(x) = x^n$ is a Sheffer set relative to the delta operator Q.
After briefly analyzing the above question, we express the delta operator Q as a sequence of real constants in the following theorem.

Theorem 2.1
For the monomial $\{x^n :\ \ n \in \mathbb{Z}^+ \cup \{0\}\}$, and for each α_r an arbitrary real constant,

$$Q(x^n) = \sum_{r=1}^{n} \binom{n}{r}\ \alpha_r\ x^{n-r}. \tag{3.1}$$

Proof.
If $n = 1$, then from the definition of delta operator, $Q(x)$ is a non zero constant.
Let it be α_1. Therefore, $Q(x) = \alpha_1 \neq 0$ and hence the result is true for $n = 1$

Let $n = 2$. Construct $Q(x^2) = c_0\ x + c_1$, by (iii) Theorem 1.5
Since Q is shift invariant, $E^aQ(x^2) = QE^a(x^2)$.

$$E^aQ(x^2) = E^a(c_0x + c_1) = c_0E^a(x) + c_1 = c_0(x + a) + c_1 = c_0x + c_0a + c_1.$$

Since $Q(a) = 0$, $Q(x) = \alpha_1$ and by Table 1, we have

$$QE^a(x^2) = Q(x + a)^2 = Q(x^2 + a^2 + 2ax) = Q(x^2) + 2aQ(x) = c_0x + c_1 + 2a\alpha_1.$$

Equating the corresponding terms in $E^aQ(x^2)$ and $QE^a(x^2)$, we get $c_0 = 2\alpha_1$
c_1 is a new independent constant which may be taken as α_2.
Hence

$$Q(x^2) = 2\alpha_1x + \alpha_2.$$

Therefore, the result is true for $n = 2$.

Let us assume that the result is true for all $n = k$.

Therefore ,

$$Q(x^k) = \sum_{r=1}^{k} \binom{k}{r}\ \alpha_r\ x^{k-r} = \binom{k}{1}\ \alpha_1\ x^{k-1} + \binom{k}{2}\ \alpha_2\ x^{k-2} + \cdots + \binom{k}{r}\ \alpha_r\ x^{k-r} + \cdots + \alpha_k \quad (3.2)$$

Since $\{x^n\}$ is a basic polynomial sequence, it satisfies $Qp_n(x) = np_{n-1}(x)$ and hence we have,

$$Q(x^k) = k\ x^{k-1} \quad (3.3)$$

From (3.3), we see that the delta operator Q is a usual derivative D.
From (3.2) and (3.3) ,

$$\binom{k}{1}\ \alpha_1\ x^{k-1} + \binom{k}{2}\ \alpha_2\ x^{k-2} + \cdots + \binom{k}{r}\ \alpha_r\ x^{k-r} + \cdots + \alpha_k = k\ x^{k-1} \quad (3.4)$$

By comparing the corresponding terms, we have $\alpha_1 = 1$ and $\alpha_j = 0,\ j = 2, 3, \cdots k$
Therefore, the result is true for $n = k$ means that

$$\alpha_1 = 1\ and\ \alpha_j = 0\ (j = 2, 3, \cdots k). \quad (3.5)$$

Now we have to show that this result is true for $n = k + 1$

$$\begin{aligned}
Q(x^{k+1}) &= Q(x^k\ x) \\
&= Q(x^k)\ x + Q(x)\ x^k \quad (for\ the\ basic\ polynomial\ sequence\ \{x^n\},\ Q\ =\ D) \\
&= \left\{\binom{k}{1}\ \alpha_1\ x^{k-1} + \binom{k}{2}\ \alpha_2\ x^{k-2} + \cdots + \binom{k}{r}\ \alpha_r\ x^{k-r} + \cdots + \alpha_k\right\}\ x + \alpha_1\ x^k \\
&= \alpha_1(k\ x^k + x^k) + \alpha_2\binom{k}{2}x^{k-1} + \alpha_3\binom{k}{3}x^{k-2} + \cdots + \alpha_k\ x \\
&= (k + 1)\ x^k\ \ by\ (3.5)
\end{aligned}$$

Thus we have

$$Q(x^{k+1}) = (k+1)\ x^k. \tag{3.6}$$

On other hand, using the property that $Qp_n(x) = n\ p_{n-1}(x)$, we have

$$Q(x^{k+1}) = (k+1)\ p_k(x) = (k+1)\ x^k. \tag{3.7}$$

From (3.6) and (3.7) , we conclude that the result is true for all $n = k+1$.

Thus we proved the Theorem (2.1). □

The following Table contains the expressions for $Q(x), Q(x^2), Q(x^3), \cdots$

Table 1. First few polynomials $Q(x^n), n = 1, 2, 3, \cdots$

n	$Q(x^n)$
1	α_1
2	$2\ \alpha_1\ x + \alpha_2$
3	$3\ \alpha_1\ x^2 + 3\ \alpha_2\ x + \alpha_3$
4	$4\ \alpha_1\ x^3 + 6\ \alpha_2\ x^2 + 4\ \alpha_3\ x + \alpha_3$
5	$5\ \alpha_1\ x^4 + 10\ \alpha_2\ x^3 + 10\ \alpha_3\ x^2 + 5\ \alpha_4\ x + \alpha_5$
6	$6\ \alpha_1\ x^5 + 15\ \alpha_2\ x^4 + 20\ \alpha_3\ x^3 + 15\ \alpha_4\ x^2 + 6\ \alpha_5\ x + \alpha_6$
7	$7\ \alpha_1\ x^6 + 21\ \alpha_2\ x^5 + 35\ \alpha_3\ x^4 + 35\ \alpha_4\ x^3 + 21\ \alpha_5\ x^2 + 7\ \alpha_6\ x^5 + \alpha_7$

The above theorem 2.1 play vital role to investigate the Characterization of delta operator for basic set as well as Sheffer set. The delta operator associated to some Sheffer polynomials such as the Laguerre, the Boole and Mittag-Leffer polynomials are inestigated in the next section.

4 Delta Operator for Sheffer Polynomials

The characterization of delta operator for the Euler, the Bernoulli and the Mott polynomials are discussed in [8]. In this section, we investigate the characterization of delta operator for the simple Laguerre polynomials, the Boole polynomials and the Mittag-Leffer polynomials.

4.1 The Laguerre Polynomials

The Laguerre polynomials are introduced by Edmond Laguerre (1834-1886). They are classified into several kinds such as Generalized Laguerre, Simple Laguerre, Associated Laguerre and Sonin polynomials. This polynomials arise in Quantum Mechanics.

The Laguerre Polynomial, usually written as $L_n(x)$, satisfies the differential equation

$$xy^{''} + (1-x)y^{'} + \lambda y = 0,\ where\ \lambda\ is\ a\ constant.$$

The Laguerre polynomial $L_n(x)$ can be defined by the generating relation

$$\tfrac{1}{1-t}\ exp\{-\tfrac{xt}{1-t}\} = \sum_{n=0}^{\infty}\ t^n\ L_n(x)$$

The Lagueree polynomials of degree n is defined by

$$L_n(x) = a_0 \sum_{r=0}^{n} (-1)^r \frac{(n!)}{(n-r)!\ (r!)^2}\ x^r.$$

Some authors define the simple Laguerre polynomial $L_n(x)$ by taking $a_0 = n!$.
That is ,

$$L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{(n!)^2}{(n-r)!\ (r!)^2}\ x^r.$$

The differential equation becomes,

$$L_n(x) = e^x\ D^n(x^n\ e^{-x}).$$

The recurrence relations are

$$(n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - xL_{n-1}(x)$$
$$xL_n^{'}(x) = nL_n(x) - nL_{n-1}(x)$$
$$L_n^{'}(x) = -\sum_{r=0}^{n-1} L_r(x).$$

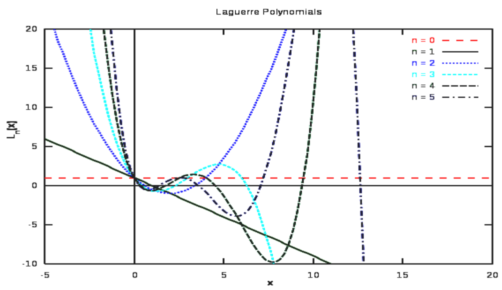

Fig 1. Laguerre polynomials

The first few Laguerre polynomials are :

$$L_0(x) = 1$$

$$L_1(x) = 1 - x$$
$$L_2(x) = 2 - 4x + x^2$$
$$L_3(x) = 6 - 18x + 9x^2 - x^3$$
$$L_4(x) = 24 - 96x + 72x^2 - 16x^3 + x^4$$

and so on.
Since $L_0(x) = 1 \neq 0$, it is a Sheffer set.
For $n = 1$, $QL_n = nL_{n-1}$ *becomes* $QL_1 = 1L_0$
From Table 1, $QL_1 = Q(1-x) = -Q(x) = -\alpha_1$ & $1L_0 = 1$
Equating the corresponding terms, we get

$$\alpha_1 = -1.$$

For $n = 2$, $QL_n = nL_{n-1}$ *becomes* $QL_2 = 2L_1$
By Table 1,

$$QL_2 = Q(2 - 4x + x^2) = -4\alpha_1 + 2\alpha_1 x + \alpha_2 \;\;\&\;\; 2L_1 = 2 - 2x$$

By Equating the corresponding terms, we get

$$\alpha_1 = -1 \; and \; \alpha_2 = -2.$$

For $n = 3$, we have $QL_3 = 3L_2$
From Table 1,

$$QL_3 = (-3\alpha_1)x^2 + (18\alpha_1 - 3\alpha_2)x - 18\alpha_1 + 9\alpha_2 - \alpha_3$$

and $3L_2 = 3x^2 - 12x + 6$
By comparing the corresponding terms, we get

$$\alpha_1 = -1, \; \alpha_2 = -2 \; and \; \alpha_3 = -6.$$

For $n = 4$, we have $QL_4 = 4L_3$.

By Table 1,

$$QL_4 = (4\alpha_1)x^3 + (6\alpha_2 - 48\alpha_1)x^2 + (4\alpha_3 - 48\alpha_2 + 144\alpha_1)x + (\alpha_4 - 16\alpha_3 + 72\alpha_2 - 96\alpha_1)$$

and also,

$$4L_3 = -4x^3 + 36x^2 - 72x + 24.$$

By equating the corresponding terms, we have

$$\alpha_1 = -1 \; \alpha_2 = -2 \; \alpha_3 = -6 \; and \; \alpha_4 = -24.$$

Applying the same procedure for $n = 5$, and $n = 6$, we get

$$\begin{aligned} \alpha_1 &= -1 = (-1)1! & \alpha_2 &= -2 = (-1)2! \\ \alpha_3 &= -6 = (-1)3! & \alpha_4 &= -24 = (-1)4! \\ \alpha_5 &= -120 = (-1)5! & \alpha_6 &= -720 = (-1)6! \end{aligned}$$

Hence we conclude that the characterization of delta operator for $L_n(x)$ is

$$\alpha_n = (-1)(n!).$$

Thus we obtain the following Proposition.

Proposition 4.1
For nth term of the Leguerre polynomials $L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{(n!)^2}{(n-r)!\,(r!)^2} x^r$, the characterization of delta operator is :

$$\alpha_r = (-1)(r!), \; for \; all \; r \geq 1. \qquad \square$$

Remark 4A: Here, $Q(x^n) = \sum_{r=1}^{n} \binom{n}{r} (-1)(r!) \, x^{n-r}$, in the view of Theorem 2.1.

4.2 Boole Polynomials

The Boole polynomials are introduced by Boole, G.(1860). It play an important role in the area of Number Theory, Algebra and Umbral Calculus. In [9], Dae San Kim and Taekyun Kim derive some new identities for the Boole polynomials from the Witt-Type formula which are related to the

Euler polynomials. Some interesting properties of the Modified Boole polynomials are studied in [10].

The generating function for the Boole polynomials is

$$\sum_{k=0}^{\infty} s_k(x;\lambda)\ \frac{t^k}{k!} = \frac{(1+t)^x}{1+(1+t)^{\lambda}}.$$

The first few Boole polynomials are

$$s_0(x;\lambda) = \frac{1}{2}$$

$$s_1(x;\lambda) = \frac{1}{4}\ (2x-\lambda)\ t$$

$$s_2(x;\lambda) = \frac{1}{4}\ [2x\ (x-\lambda-1)+\lambda]$$

and so on.

Jordon (1965) considers the another form of Boole polynomials $r_n(x)$ which also a Sheffer sequence.

These polynomials have the following generating function

$$\sum_{k=0}^{\infty} r_n(x)\ \frac{t^k}{k!} = \frac{2(1+t)^x}{2+t}$$

The first few are

$$r_0(x) = 1$$

$$r_1(x) = \frac{1}{2}\ (2x-1)$$

$$r_2(x) = \frac{1}{2}\ (2x^2-4x+1)$$

$$r_3(x) = \frac{1}{4}\ (4x^3-18x^2+20x-3)$$

and so on.

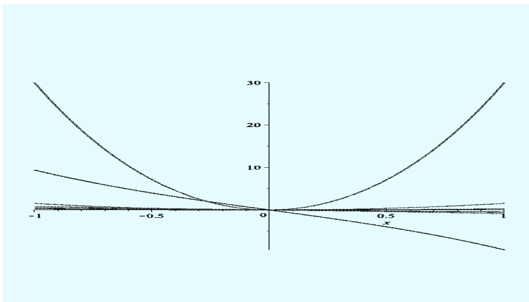

Fig 2. Boole polynomials

For $n = 1$,

$$Qr_n = nr_{n-1}\ \textit{becomes}\ Qr_1 = 1r_0$$

From Table 1,

$$Qr_1 = \alpha_1 \ and \ 1r_0 = 1 \Rightarrow \alpha_1 = 1.$$

For $n = 2$,

$$Qr_n = nr_{n-1} \ becomes \ Qr_2 = 2r_1$$

By Table 1,

$$Qr_2 = 2\alpha_1 x + \alpha_2 - 2\alpha_1 \quad and \ 2r_1 = 2x - 1 \Rightarrow \alpha_1 = 1 \ and \ \alpha_2 = 1.$$

For $n = 3$,

$$Qr_n = nr_{n-1} \ becomes \ Qr_3 = 3r_2$$

From Table 1,

$$Qr_3 = (3\alpha_1)x^2 + (3\alpha_2 - 9\alpha_1)x + \alpha_3 - \frac{9}{2}\alpha_2 + 5\alpha_1 \quad and \quad 3r_2 = 3x^2 - 6x + \frac{3}{2}.$$

Equating the corresponding terms, we get

$$\alpha_1 = 1, \quad \alpha_2 = 1 \ and \ \alpha_3 = 1.$$

By Similar procedure, we get

$$\alpha_r = 1 \quad for \ all \ r \geq 1.$$

Hence the characterization of the delta operator for Boole polynomials $r_n(x)$, being $\alpha_r = 1$, for all $r \geq 1$.

Thus we obtain the following Proposition.

Proposition 4.2
For the Boole polynomials $r_n(x)$, the characterization of delta operator is :

$$\alpha_r = 1, \ for \ all \ r \geq 1. \quad \square$$

Remark 4B: Here, $Q(x^n) = \sum_{r=0}^{n} \binom{n}{r} x^{n-r}$

4.3 Mittag-Leffer Polynomials

The Mittag-Leffer polynomials are studied by Mittag-Leffer(1891). It is a study of the analytical representation of the integrals and invariants of a linear homogeneous differential equations. The same polynomial is studied by H. Bateman [11]. Recently, Some new and explicit identities for Mittag-Leffer polynomials are derived by Miomir et.al [12] and the behaviour of Mittag-Leffer polynomials are analyzed by Jordanka and Paneva-Konovska [13].

The generating function for the Mittag-Leffer polynomial is

$$\sum_{k=0}^{\infty} \frac{M_k(x)}{k!} \ t^k = \left\{ \frac{1+t}{1-t} \right\}^x .$$

The Mittag-Leffer polynomial is defined by

$$M_n(x) = \sum_{k=0}^{n} \binom{n}{k} (n-1)_{n-k} \ 2^k \ [x]_k.$$

Here, $[x]_k$ is a falling factorial.

The binomial identity associated with this Mittag-Leffer polynomials is

$$M_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} M_k(x)\ M_{n-k}(y).$$

The Mittag-Leffer polynomials satisfy the following recurrence formula

$$M_{n+1}(x) = \frac{1}{2}\ x\ [M_n(x+1) + 2M_n(x) + M_n(x-1)].$$

The first few Mittag-Leffer polynomials are

$$M_0(x) = 1$$
$$M_1(x) = 2x$$
$$M_2(x) = 4x^2$$
$$M_3(x) = 8x^3 + 4x$$
$$M_4(x) = 16x^4 + 32x^2$$
$$M_5(x) = 32x^5 + 160x^3 + 48x$$

and so on.

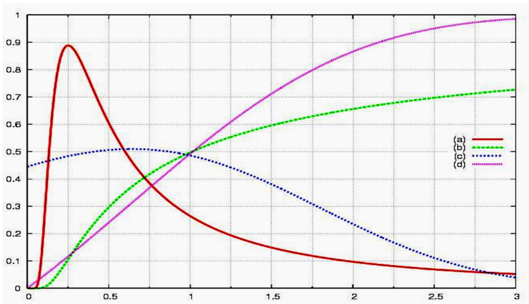

Fig 3. Mittag-Leffer polynomials

For $n = 1$, $QM_n = nM_{n-1}$ becomes $QM_1 = 1M_0$
From Table 1,

$$QM_1 = 2\ \alpha_1 \quad and \quad 1M_0 = 1\ \Rightarrow \alpha_1 = \frac{1}{2}.$$

For $n = 2$, $QM_n = nM_{n-1}$ becomes $QM_2 = 2M_1$
By Table 1,

$$QM_2 = 8\alpha_1 x + 4\alpha_2 \quad and \quad 2M_1 = 4x\ \Rightarrow \alpha_1 = \frac{1}{2}\ \ \&\ \ \alpha_2 = 0.$$

For $n = 3$, $QM_n = nM_{n-1}$ becomes $QM_3 = 3M_2$
From Table 1,

$$QM_3 = 24\alpha_1 x^2 + 24\alpha_2 x + 8\alpha_3 + 4\alpha_1 \quad and \quad 3M_2 = 12x^2$$

Equating the corresponding terms, we get

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = 0\ \ \&\ \ \alpha_3 = -\frac{1}{4}.$$

For $n = 4$, $QM_n = nM_{n-1}$ becomes $QM_4 = 4M_3$
By Table 1,

$$QM_4 = 64\alpha_1 x^3 + 96\alpha_2 x^2 + (64\alpha_3 + 64\alpha_1)x + 16\alpha_4 + 32\alpha_2)$$

and

$$4M_3 = 32x^3 + 16x$$

Comparing the corresponding terms, we get

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = 0 \quad \alpha_3 = -\frac{1}{4} \quad \& \quad \alpha_4 = 0.$$

For $n = 5$, $\quad QM_n = nM_{n-1}$ becomes $QM_5 = 5M_4$
By Table 1

$$QM_5 = 160\alpha_1 x^4 + 320\alpha_2 x^3 + (320\alpha_3 + 480\alpha_1)x^2 + (160\alpha_4 + 480\alpha_2)x + 32\alpha_5 + 160\alpha_3 + 48\alpha_1$$

and

$$5M_4 = 80x^4 + 160x^2$$

Comparing the corresponding terms, we get

$$\alpha_1 = \frac{1}{2}, \quad \alpha_2 = 0 \quad \alpha_3 = -\frac{1}{4} \quad \alpha_4 = 0 \quad \& \quad \alpha_5 = \frac{1}{2}.$$

Applying the same procedure for $n = 6$, $n = 7$ and $n = 8$, we get

$$\begin{aligned} \alpha_1 &= \frac{1}{2} & \alpha_2 &= 0 \\ \alpha_3 &= -\frac{1}{4} & \alpha_4 &= 0 \\ \alpha_5 &= \frac{1}{2} & \alpha_6 &= 0 \\ \alpha_7 &= -\frac{1}{4} & \alpha_8 &= 0 \end{aligned}$$

Thus we have the following Proposition,

Proposition 4.3
For the Mittag-Leffer polynomials $M_n(x) = \sum_{k=0}^{n} \binom{n}{k} (n-1)_{n-k}\, 2^k\, [x]_k$, the characterization of delta operator is:

$$\alpha_r = \begin{cases} 0 & \text{if } r \text{ is even} \\ \frac{1}{2} & \text{if } r = 1, 5, 9, \cdots \\ -\frac{1}{4} & \text{if } r = 3, 7, 11, \cdots \end{cases}$$

The Mittag-Leffer polynomials $M_n(x)$ are related to Pidduck polynomials $P_n(x)$ by

$$P_n(x) = \frac{1}{2}\,(e^t + 1)\, M_n(x).$$

The characterization of the delta operator for Pidduck polynomials is same that of the characterization of delta operator for Mittag-Leffer polynomials.

All the results are shown vividly in the following Table.

Table 2. Characterization of Delta Operator for given Sheffer Set

Polynomials	Characterization of Delta Operator
Laguerre	$\alpha_r = (-1)(r!),\ \ r \geq 1.$
Boole	$\alpha_r = 1$ for all r $\geq 1.$
Mittag-Leffer	$\alpha_r = 0$ if r is even $\alpha_r = \frac{1}{2}$ if $r = 1, 5, 9 \cdots.$ $\alpha_r = -\frac{1}{4}$ if $r = 3, 7, 11 \cdots$

Thus having a study of Sheffer polynomials through Finite Operator Calculus in which the delta operator is found and analyzed for any given set of Sheffer polynomials through the sequential representation of delta operator.

5 Conclusion and Further Directions

Usually the special polynomials are studied through the differential equations, the generating functions and the recurrence relations. Now a way is opened to study the special functions in particular, special polynomials through the sequential representation of delta operator in Finite Operator Calculus. K.K. Velukutty [14] introduced a new difference operator and established a Discrete Analytic Fuction Theory namely q-monodiffric theory. Moreover, A.K.Kwasniewski [15] proposed Finite Operator q-Calculus by using q-delta operator and q-basic polynomial sequence. Hence this study may be extended to the investigation of q-delta operator for q-Sheffer polynomials.

Competing Interests

Authors have declared that no competing interests exist.

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