Asian Research Journal of Mathematics
1(5): 1-16, 2016; Article no.ARJOM.28960

SCIENCEDOMAIN *international*
www.sciencedomain.org

SCIENCEDOMAIN

Common Fixed Point Results in Ordered S-metric Spaces for Rational Type Expressions

Arvind Bohre[1], Suresh Nagle[2*], Rashmi Jain[3] and Manoj Ughade[4]

[1]Department of Mathematics, Faculty of Science, Government Girls P.G. College, Sagar, MP, India.
[2]Research Scholar, Department of Mathematics, Atal Bihari Vajpai Hindi University, Bhopal, MP, India.
[3]Research Scholar, Department of Mathematics, J H Government Post Graduate College, Betul, MP, India.
[4]Department of Mathematics, Faculty of Science, Sarvepalli Radhakrishnan University, Bhopal, MP, India.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2016/28960
Editor(s):
(1) Sheng Zhang, School of Mathematics and Physics, Bohai University, Jinzhou, China.
Reviewers:
(1) Xiaolan Liu, Sichuan University of Science and Engineering, China.
(2) Muhammad Nazam, International Islamic University, Pakistan.
Complete Peer review History: http://www.sciencedomain.org/review-history/17065

Original Research Article

Received: 15th August 2016
Accepted: 24th September 2016
Published: 29th November 2016

Abstract

The aim of this paper is to present some common fixed point theorems for g-monotone maps involving rational expression in the framework of S-metric spaces endowed with a partial order using a class of pairs of functions satisfying certain assumptions.

Keywords: Common fixed point; S-metric space; contractions; partially ordered set, altering distance function.

1 Introduction and Preliminaries

Metric spaces are very important in mathematics and applied sciences. So, some authors have tried to give generalizations of metric spaces in several ways. For example, Gahler [1] and Dhage [2] introduced the concepts of 2-metric spaces and *D*-metric spaces, respectively. In 2006, Mustafa and Sims [3] introduced a

**Corresponding author: E-mail: snghelpyou@gmail.com;*

new structure of generalized metric spaces which are called *G*-metric spaces as a generalization of metric spaces (X, d) to develop and introduce a new fixed point theory for various mappings in this new structure. Sedghi et al. [4] introduced the notion of a D^*-metric space.

Das and Gupta [5] proved the following fixed point theorem.

Theorem 1.1: (see [5]) Let (X, d) be a complete metric space and $f: X \to X$ a mapping such that there exist $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ satisfying

$$d(fx, fy) \leq \alpha \frac{d(y,fy)[1 + d(x,fx)]}{1 + d(x,y)} + \beta d(x, y) \tag{1.1}$$

for all $x, y \in X$. Then f has a unique fixed point in X.

For more details on fixed point results with rational expressions, see [6-8].

Cabrera et al. [9] proved Theorem 1.1 in the context of partially ordered metric spaces.

Definition 1.2 (see [9]) Let $(X, \preceq)$ is a partially ordered set and $f : X \to X$ is said to be monotone non-decreasing if for all $x, y \in X$,

$$x \preceq y \;\Rightarrow\; fx \preceq fy. \tag{1.2}$$

Theorem 1.3: (see [9]) Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist a metric d on X such that (X, d) be a complete metric space. Let $f: X \to X$ be a continuous and non-decreasing mapping such that (1.1) is satisfied for all $x, y \in X$ with $x \preceq y$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Theorem 1.4: (see [9]) Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist a metric d on X such that (X, d) be a complete metric space. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$, for all $n \in \mathbb{N}$. Let $f: X \to X$ be a non-decreasing mapping such that (1.1) is satisfied for all $x, y \in X$ with $x \preceq y$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

Theorem 1.5: (see [9]) In addition to the hypothesis of Theorem 1.3 or Theorem 1.4, suppose that for every $x, y \in X$, there exist $u \in X$ such that $u \preceq x$ and $u \preceq y$. Then T has a unique fixed point.

In this paper, we establish some common fixed point theorems for g-monotone mappings involving rational expression in the framework of S-metric spaces endowed with a partial order using a class of pairs of functions satisfying certain assumptions.

Sedghi et al. [10] introduced a new generalized metric space called an S-metric space.

Definition 1.6: (see [10]) Let X be a non-empty set. An S-metric on X is a function $S: X^3 \to [0, +\infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,

(S1). $S(x, y, z) \geq 0$;
(S2). $S(x, y, z) = 0$ if and only if $x = y = z$;
(S3). $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

Then S is called an S-metric on X and (X, S) is called an S-metric space.

The following is the intuitive geometric example for S-metric spaces.

Example 1.7: (see [10], Example 2.4) Let $X = \mathbb{R}^2$ and d be the ordinary metric on X. Put $S(x,y,z) = d(x,y) + d(x,z) + d(y,z)$ for all $x, y \in \mathbb{R}^2$, that is, S is the perimeter of the triangle given by x, y, z. Then S is an S-metric on X.

Lemma 1.8: (see [10], Lemma 2.5) Let (X, S) be an S-metric space. Then $S(x,x,y) = S(y,y,x)$ for all $x, y \in X$.

Lemma 1.9: (see [11], Lemma 1.6) Let (X, S) be an S-metric space. Then $S(x,x,z) \leq 2S(x,x,y) + S(y,y,z)$ and $S(x,x,z) \leq 2S(x,x,y) + S(z,z,y)$ for all $x, y, z \in X$.

Definition 1.10: (see [10]) Let X be an S-metric space.

(i). A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ converges to x if and only if $lim_{n\to\infty} S(x_n, x_n, x) = 0$. That is for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0, S(x_n, x_n, x) < \epsilon$ and we denote this by $lim_{n\to\infty} x_n = x$.
(ii). A sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is called a Cauchy if $lim_{n,m\to\infty} S(x_n, x_n, x_m) = 0$. That is, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $, m \geq n_0,\ S(x_n, x_n, x_m) < \epsilon$.
(iii). X is called complete if every Cauchy sequence in X is a convergent sequence.

From (see [10], Examples in page 260), we have the following.

Example 1.11:

(a). Let $\mathbb{R}$ be the real line. Then $S(x,y,z) = |x - z| + |y - z|$ for all $x, y, z \in \mathbb{R}$, is an S-metric on $\mathbb{R}$. This S-metric is called the usual S-metric on $\mathbb{R}$. Furthermore, the usual S-metric space $\mathbb{R}$ is complete.
(b). Let Y be a non-empty set of $\mathbb{R}$. Then $S(x,y,z) = |x - z| + |y - z|$ for all $x, y, z \in Y$, is an S-metric on Y. If Y is a closed subset of the usual metric space $\mathbb{R}$, then the S-metric space Y is complete.

Lemma 1.12: (see [10], Lemma 2.11) Let (X, S) be an S-metric space. If the sequence $\{x_n\}_{n=1}^{\infty}$ in X converges to x, then x is unique.

Lemma 1.13: (see [10], Lemma 2.12) Let (X, S) be an S-metric space. If $lim_{n\to+\infty} x_n = x$ and $lim_{n\to+\infty} y_n = y$, then $lim_{n\to+\infty} S(x_n, x_n, y_n) = S(x,x,y)$.

Remark 1.14: (see [11]) It is easy to see that every D*-metric (see [4]) is S-metric, but in general the converse is not true, see the following example.

Example 1.15: (see [11]) Let $X = \mathbb{R}^n$ and $\|.\|$ a norm on X, then $S(x,y,z) = \|y + z - 2x\| + \|y - z\|$ is S-metric on X, but it is not D*-metric because it is not symmetric.

The following lemma shows that every metric space is an S-metric space.

Lemma 1.16: (see [11], Lemma 1.10) Let (X, d) be a metric space. Then we have

1. $S_d(x,y,z) = d(x,z) + d(y,z)$ for all $x, y, z \in X$ is an S-metric on X.
2. $lim_{n\to+\infty} x_n = x$ in (X, d) if and only if $lim_{n\to+\infty} x_n = x$ in (X, S_d).
3. $\{x_n\}_{n=1}^{\infty}$ is Cauchy in (X, d) if and only if $\{x_n\}_{n=1}^{\infty}$ is Cauchy in (X, S_d).
4. (X, d) is complete if and only if (X, S_d) is complete.

In 2012, Sedghi et al. [10] asserted that an S-metric is a generalization of a G-metric, that is, every G-metric is an S-metric, see [10, Remarks 1.3] and [10, Remarks 2.2]. The Example 2.1 and Example 2.2 of Dung et al. [12] shows that this assertion is not correct. Moreover, the class of all S-metrics and the class of all G-metrics are distinct. For more results on S-metric spaces, see [11-12].

In this paper, we consider the following class of pairs of functions $\mathfrak{F}$.

Definition 1.17: (see [13]) A pair of functions (φ,ϕ) is said to belong to the class $\mathfrak{F}$, if they satisfy the following conditions:

(b1). $\varphi,\phi: [0,\infty) \to [0,\infty)$;
(b2). for $t,s \in [0,\infty)$, $\varphi(t) \le \phi(s)$ then $t \le s$;
(b3). for $\{t_n\}$ and $\{s_n\}$ sequence in $[0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = a$, if $\varphi(t_n) \le \phi(s_n)$ for any $n \in \mathbb{N}$, then $a = 0$.

Remark 1.18: (see [13], Remark 4) Note that, if $(\varphi,\phi) \in \mathfrak{F}$ and $\varphi(t) \le \phi(t)$, then $t = 0$, since we can take $t_n = s_n = t$ for any $n \in \mathbb{N}$ and by (b3) we deduce that $t = 0$.

Example 1.19: (see [13], Example 5) Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous and increasing function such that $\varphi(t) = 0$ if and only if $t = 0$ (these functions are known in the literature as altering distance functions). Let $\phi : [0,\infty) \to [0,\infty)$ be a non-decreasing function such that $\phi(t) = 0$ if and only if $t = 0$ and suppose that $\phi \le \varphi$. Then the pair $(\varphi, \varphi - \phi) \in \mathfrak{F}$.

An interesting particular case is when φ is the identity mapping, $\varphi = 1_{[0,\infty)}$ and $\phi: [0,\infty) \to [0,\infty)$ is a non-decreasing function such that $\phi(t) = 0$ if and only if $t = 0$ and $\phi(t) \le t$ for any $t \in [0,\infty)$.

Example 1.20: (see [13], Example 6) Let S be the class of functions defined by

$$S = \{\alpha : [0,\infty) \to [0,1) : \{\alpha(t_n) \to 1 \Rightarrow t_n \to 0\}\}.$$

Let us consider the pairs of functions $(1_{[0,\infty)}, \alpha 1_{[0,\infty)})$, where $\alpha \in S$ and $\alpha 1_{[0,\infty)}$ is defined by $\left(\alpha 1_{[0,\infty)}\right)(t) = \alpha(t)t$, for $t \in [0,\infty)$. Then $(1_{[0,\infty)}, \alpha 1_{[0,\infty)}) \in \mathfrak{F}$.

Remark 1.21: (see [13], Remark 7) Suppose that $g : [0,\infty) \to [0,\infty)$ is an increasing function and $(\varphi,\phi) \in \mathfrak{F}$. Then it is easily seen that the pair $(g \circ \varphi, g \circ \phi) \in \mathfrak{F}$.

For more fixed point results with alternating distance function, see [14-19].

Definition 1.22: (see [20]) Let $(X,\preceq)$ be a partially ordered set and let $f,g: X \to X$ be two maps. Map f is called g-non-decreasing if $gx \preceq gy$ implies $fx \preceq fy$ for all $x,y \in X$.

Definition 1.23: (see [21]) Let X be a non-empty set and let $f,g: X \to X$ be two maps. f and g are called to commute at $x \in X$ if $f(gx) = g(fx)$.

2 Main Results

In this section, we investigate the common fixed point problem on S-metric spaces. The following result states the existence of a common fixed point of two maps f and g on partially ordered S-metric spaces.

Theorem 2.1: Let $(X,\preceq)$ is a partially ordered set. Suppose that there exists an S-metric S on X such that (X,S) be a complete S-metric space. Let $f,g: X \to X$ be two maps with $f(X) \subset g(X)$, f is g-non-decreasing map and $g(X)$ is closed such that there exists a pair of functions $(\varphi,\phi) \in \mathfrak{F}$ satisfying

$$\varphi\big(S(fx,fx,fy)\big) \le max\left\{\phi\big(S(gx,gx,gy)\big), \phi\left(\frac{S(gy,gy,fy)[1+S(gx,gx,fx)]}{1+S(fx,fx,fy)}\right)\right\}, \tag{2.1}$$

for all $x, y \in X$ with $gx \preceq gy$. Assume that if $\{gx_n\}$ is non-decreasing sequence in X such that $gx_n \to gu$, then $gx_n \preceq gu \preceq g(gu)$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point. Furthermore, if f and g commute at the coincidence point, then f and g have a common fixed point.

Proof Since $f(X) \subset g(X)$, we can choose $x_1 \in X$ such that $gx_1 = fx_0$. Again, from $f(X) \subset g(X)$ we can choose $x_2 \in X$ such that $gx_2 = fx_1$. Continuing this process, we can choose a sequence $\{x_n\}$ in X such that

$$gx_{n+1} = fx_n, \forall\, n \in \mathbb{N}. \tag{2.2}$$

Since $gx_0 \preceq fx_0$ and $gx_1 = fx_0$, we have $gx_0 \preceq gx_1$. Since f is g-non-decreasing, we get $fx_0 \preceq fx_1$. By using (2.2), we have $gx_1 \preceq gx_2$. Again, since f is g-non-decreasing, we get $fx_1 \preceq fx_2$, that is, $gx_2 \preceq gx_3$. Continuing this process, we obtain

$$fx_n \preceq fx_{n+1}, gx_{n+1} \preceq gx_{n+2}, \forall\, n \in \mathbb{N}$$

Denote $\delta_n = S(fx_n, fx_n, fx_{n+1}), \forall\, n \in \mathbb{N}$. To prove that f and g have a coincidence point. We consider two following cases.

Case 1. There exists n_0 such that $\delta_{n_0} = 0$. It implies that $x_{n_0} = fx_{n_0+1}$. By (2.2), we get $fx_{n_0+1} = gx_{n_0+1}$. Therefore, x_{n_0+1} is a coincidence point of f and g.

Case 2. Let $\delta_n > 0$ for all $n \in \mathbb{N}$. We will show that $\lim_{n\to\infty} \delta_n = 0$. Since $fx_{n-1} \prec fx_n$ for all $n \geq 1$, applying the contractive condition (2.1), we have

$$\begin{aligned}\varphi(\delta_n) &= \varphi(S(fx_n, fx_n, fx_{n+1})) \\ &\leq \max\left\{\phi\big(S(gx_n, gx_n, gx_{n+1})\big), \phi\left(\frac{S(gx_{n+1}, gx_{n+1}, fx_{n+1})[1+S(gx_n, gx_n, fx_n)]}{1+S(fx_n, fx_n, fx_{n+1})}\right)\right\} \\ &= \max\left\{\phi\big(S(fx_{n-1}, fx_{n-1}, fx_n)\big), \phi\left(\frac{S(fx_n, fx_n, fx_{n+1})[1+S(fx_{n-1}, fx_{n-1}, fx_n)]}{1+S(fx_n, fx_n, fx_{n+1})}\right)\right\} \\ &= \max\left\{\phi(\delta_{n-1}), \phi\left(\frac{\delta_n[1+\delta_{n-1}]}{1+\delta_n}\right)\right\}\end{aligned} \tag{2.3}$$

Now, we consider two following subcases.

Subcase 1. Consider

$$\max\left\{\phi(\delta_{n-1}), \phi\left(\frac{\delta_n[1+\delta_{n-1}]}{1+\delta_n}\right)\right\} = \phi(\delta_{n-1}) \tag{2.4}$$

In this case from (2.3), we have

$$\varphi(\delta_n) \leq \phi(\delta_{n-1}) \tag{2.5}$$

Since $(\varphi, \phi) \in \mathfrak{F}$, we deduce that $\delta_n \leq \delta_{n-1}$.

Subcase 2. If

$$\max\left\{\phi(\delta_{n-1}), \phi\left(\frac{\delta_n[1+\delta_{n-1}]}{1+\delta_n}\right)\right\} = \phi\left(\frac{\delta_n[1+\delta_{n-1}]}{1+\delta_n}\right) \tag{2.6}$$

In this case from (2.3), we have

$$\varphi(\delta_n) \leq \phi\left(\frac{\delta_n[1+\delta_{n-1}]}{1+\delta_n}\right) \tag{2.7}$$

Since $(\varphi, \phi) \in \mathfrak{F}$ and $\delta_n > 0$, we deduce that $\delta_n \leq \delta_{n-1}$.

The conclusions of two above subcases,

$$\delta_n \leq \delta_{n-1} \tag{2.8}$$

It follows from (2.8) that the sequence $\{\delta_n\}$ of real numbers is monotone decreasing.Then there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = r. \tag{2.9}$$

Now, we shall show that $r = 0$.

Denote
$A = \{n \in \mathbb{N} : n \text{ satisfies } (2.4)\}$ and $B = \{n \in \mathbb{N} : n \text{ satisfies } (2.6)\}$.

From (2.3), we have $Card\ A = \infty$ or $Card\ B = \infty$. Let us suppose that $Card\ A = \infty$. Then from (2.3), we can find infinitely natural numbers n satisfying inequality (2.5) and since $(\varphi, \phi) \in \mathfrak{F}$, we infer from (2.9) and condition (b3) that $r = 0$. On the other hand, if $Card\ B = \infty$,, then from (2.3), we can find infinitely many $n \in \mathbb{N}$ satisfying inequality (2.7). Since $(\varphi, \phi) \in \mathfrak{F}$, we obtain

$$\delta_n \leq \frac{\delta_n[1+\delta_{n-1}]}{1+\delta_n}$$

for infinitely many $n \in \mathbb{N}$. Letting the limit as $n \to \infty$ and taking into account that (2.9), we deduce that $r \leq r\,(1+r)/(1+r)$ and consequently, we obtain $r = 0$.

Therefore

$$\lim_{n\to\infty} \delta_n = r = 0. \tag{2.10}$$

Now, we will show that $\{fx_n\}$ is a Cauchy sequence. Suppose on the contrary that $\{fx_n\}$ is not a Cauchy sequence. Then given $\epsilon > 0$, we will construct a pair of subsequences $\{fx_{m_i}\}$ and $\{fx_{n_i}\}$ violating the following condition for least integer m_i such that $m_i > n_i > i$, where $i \in \mathbb{N}$:

$$\gamma_i = S\left(fx_{n_i}, fx_{n_i}, fx_{m_i}\right) \geq \epsilon \tag{2.11}$$

In addition, upon choosing the smallest possible m_i, we may assume that

$$S\left(x_{n_i}, x_{n_i}, x_{m_i-1}\right) < \epsilon \tag{2.12}$$

From Lemma 1.1, Lemma 1.2, (2.11) and (2.12), we have

$$\begin{aligned}
\epsilon &\leq \gamma_i \\
&= S\left(fx_{n_i}, fx_{n_i}, fx_{m_i}\right) \\
&= S\left(fx_{m_i}, fx_{m_i}, fx_{n_i}\right) \\
&\leq 2S\left(fx_{m_i}, fx_{m_i}, fx_{m_i-1}\right) + S\left(fx_{n_i}, fx_{n_i}, fx_{m_i-1}\right) \\
&\leq 2S\left(fx_{m_i-1}, fx_{m_i-1}, fx_{m_i}\right) + S\left(fx_{n_i}, fx_{n_i}, fx_{m_i-1}\right) \\
&\leq \epsilon + 2\delta_{m_i-1}
\end{aligned} \tag{2.13}$$

On letting the limit as $i \to \infty$ in the above inequality, we obtain

$$lim_{i\to\infty} \gamma_i = \epsilon \tag{2.14}$$

If we denote $\beta_i = S(fx_{n_i+1}, fx_{n_i+1}, fx_{m_i+1})$, we notice that

$$\begin{aligned}|\beta_i - \gamma_i| &= |S(fx_{n_i+1}, fx_{n_i+1}, fx_{m_i+1}) - \gamma_i|\\ &\le 2S(fx_{n_i+1}, fx_{n_i+1}, fx_{n_i}) + S(fx_{m_i+1}, fx_{m_i+1}, fx_{n_i}) - \gamma_i\\ &= 2S(fx_{n_i}, fx_{n_i}, fx_{n_i+1}) + 2S(fx_{m_i+1}, fx_{m_i+1}, fx_{m_i}) - \gamma_i\\ &\le 2\delta_{n_i} + 2S(fx_{m_i+1}, fx_{m_i+1}, fx_{m_i}) + S(fx_{n_i}, fx_{n_i}, fx_{m_i}) - \gamma_i\\ &= 2\delta_{n_i} + 2S(fx_{m_i}, fx_{m_i}, fx_{m_i+1}) + \gamma_i - \gamma_i\\ &= 2\delta_{n_i} + 2\delta_{m_i}\end{aligned} \tag{2.15}$$

On making $i \to \infty$, we immediately obtain that:

$$lim_{i\to\infty} \beta_i = \epsilon \tag{2.16}$$

It follows from (2.2) and (2.3) that $gx_{n_i+1} = fx_{n_i} \preceq fx_{m_i} = gx_{m_i+1}$.Now using contractive condition (2.1), we get

$$\begin{aligned}\varphi(\beta_i) &= \varphi\left(S(fx_{n_i+1}, fx_{n_i+1}, fx_{m_i+1})\right)\\ &\le max\left\{\phi\left(S(gx_{n_i+1}, gx_{n_i+1}, gx_{m_i+1})\right), \phi\left(\frac{S(gx_{m_i+1}, gx_{m_i+1}, fx_{m_i+1})[1+S(gx_{n_i+1}, gx_{n_i+1}, fx_{n_i+1})]}{1+S(fx_{n_i+1}, fx_{n_i+1}, fx_{m_i+1})}\right)\right\}\\ &= max\left\{\phi\left(S(fx_{n_i}, fx_{n_i}, fx_{m_i})\right), \phi\left(\frac{S(fx_{m_i}, fx_{m_i}, fx_{m_i+1})[1+S(fx_{n_i}, fx_{n_i}, fx_{n_i+1})]}{1+S(fx_{n_i+1}, fx_{n_i+1}, fx_{m_i+1})}\right)\right\}\\ &= max\left\{\phi(\gamma_i), \phi\left(\frac{\delta_{m_i}[1+\delta_{n_i}]}{1+\beta_i}\right)\right\}\end{aligned} \tag{2.17}$$

Let us put

$$B = \{i \in \mathbb{N} : \varphi(\beta_i) \le \phi(\gamma_i)\},$$

$$C = \left\{i \in \mathbb{N} : \varphi(\beta_i) \le \phi\left(\frac{\delta_{m_i}[1+\delta_{n_i}]}{1+\beta_i}\right)\right\}.$$

By (2.17), we have $Card\ B = \infty$ or $Card\ C = \infty$. Let us suppose that $Card\ B = \infty$. Then there exists infinitely many $i \in \mathbb{N}$ satisfying inequality $\varphi(\beta_i) \le \phi(\gamma_i)$ and since $(\varphi, \phi) \in \mathfrak{F}$, we have by letting the limit as $i \to \infty$, $\lim_{i\to\infty} \beta_i \le \lim_{i\to\infty} \gamma_i$. We infer from (2.14) and (2.16) that $\epsilon = 0$. This is a contradiction.

On the other hand, if $Card\ C = \infty$, then we can find infinitely many $i \in \mathbb{N}$ satisfying inequality $\varphi(\beta_i) \le \phi\left(\frac{\delta_{m_i}[1+\delta_{n_i}]}{1+\beta_i}\right)$ and since $(\varphi, \phi) \in \mathfrak{F}$, we obtain $\beta_i \le \frac{\delta_{m_i}[1+\delta_{n_i}]}{1+\beta_i}$. Om letting the limit as $i \to \infty$ and using

(2.10) and (2.16) we get $\epsilon \leq 0$, which is a contradiction. Therefore, since in both possibilities $Card\ B = \infty$ and $Card\ C = \infty$, we obtain a contradiction, we deduce that $\{fx_n\}$ is a Cauchy sequence. From (2.1), we have $\{gx_{n+1}\}$ is also a Cauchy sequence. Since $g(X)$ is closed, there exists $u \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu. \tag{2.18}$$

Now we will show that u is a coincidence point of f and g. Since $\{gx_n\}$ is non-decreasing sequence in X such that $gx_n \to gu$, then $gx_n \preceq gu$ for all $n \in \mathbb{N}$. Applying contractive condition (2.1), we obtain for any $n \in \mathbb{N}$,

$$\varphi\big(S(fu,fu,fx_n)\big) \leq max\left\{\phi\big(S(gu,gu,gx_n)\big), \phi\left(\frac{S(gx_n,gx_n,fx_n)[1+S(gu,gu,fu)]}{1+S(fu,fu,fx_n)}\right)\right\} \tag{2.19}$$

Put

$$E = \left\{n \in \mathbb{N} : \varphi\big(S(fu,fu,fx_n)\big) \leq \phi\big(S(gu,gu,gx_n)\big)\right\},$$

$$F = \left\{n \in \mathbb{N} : \varphi\big(S(fu,fu,fx_n)\big) \leq \phi\left(\frac{S(gx_n,gx_n,fx_n)[1+S(gu,gu,fu)]}{1+S(fu,fu,fx_n)}\right)\right\}.$$

By (2.19), we have $Card\ E = \infty$ or $Card\ F = \infty$. Let us suppose that $Card\ E = \infty$. Then there exists infinitely many $n \in \mathbb{N}$ satisfying inequality $\varphi\big(S(fu,fu,fx_n)\big) \leq \phi\big(S(gu,gu,gx_n)\big)$ and since $(\varphi,\phi) \in \mathfrak{F}$, letting the limit as $n \to \infty$ and using (2.18), we obtain $\lim_{n\to\infty} S(fu,fu,fx_n) = 0$, and consequently, we obtain $\lim_{n\to\infty} fx_n = fu$. The uniqueness of the limit, since $\lim_{n\to\infty} fx_n = gu$, we have $fu = gu$.

On the other hand, if $Card\ F = \infty$, we can find infinitely many $n \in \mathbb{N}$ satisfying inequality

$$\varphi\big(S(fu,fu,fx_n)\big) \leq \phi\left(\frac{S(gx_n,gx_n,fx_n)[1+S(gu,gu,fu)]}{1+S(fu,fu,fx_n)}\right) \tag{2.20}$$

Now, passing to the limit in

$$S(gx_n,gx_n,fx_n) \leq S(gx_n,gx_n,gu) + S(gx_n,gx_n,gu) + S(fx_n,fx_n,gu)$$

as $n \to \infty$, we obtain $\lim_{n\to\infty} S(gx_n,gx_n,fx_n) = 0$. Since $(\varphi,\phi) \in \mathfrak{F}$, letting the limit as $n \to \infty$ in (2.20) and taking into account that $\lim_{n\to\infty} S(gx_n,gx_n,fx_n) = 0$, we deduce that $\lim_{n\to\infty} S(fu,fu,fx_n) = 0$ and consequently, we obtain $\lim_{n\to\infty} fx_n = fu$. Thus, we have $fu = gu$. Therefore, in both the cases, u is a coincidence point of f and g.

Furthermore, we will show that gu is a common fixed point of f and g if f and g are commutative at the coincidence point. Indeed, we have $f(gu) = g(fu) = g(gu)$. By (2.3) and (2.18), we have $gu \preceq g(gu)$. Applying contractive condition (2.1), we obtain

$$\varphi\Big(S\big(fu,fu,f(gu)\big)\Big) \leq max\left\{\phi\big(S(gu,gu,g(gu))\big), \phi\left(\frac{S(g(gu),g(gu),f(gu))[1+S(gu,gu,fu)]}{1+S(fu,fu,f(gu))}\right)\right\}$$

$$= max\left\{\phi\big(S(gu,gu,g(gu))\big), \phi\left(\frac{S(g(gu),g(gu),f(gu))}{1+S(fu,fu,f(gu))}\right)\right\} \tag{2.21}$$

Consider

$$max\left\{\phi\Big(S\big(gu,gu,g(gu)\big)\Big), \phi\left(\frac{S\big(g(gu),g(gu),f(gu)\big)}{1+S\big(fu,fu,f(gu)\big)}\right)\right\} = \phi\big(S(gu,gu,g(gu))\big)$$

In this case, from (2.21) we have $\varphi\left(S\big(fu,fu,f(gu)\big)\right) \le \phi\big(S(gu,gu,g(gu))\big)$. Now, since$(\varphi,\phi) \in \mathfrak{F}$, and using $f(gu) = g(fu) = g(gu)$, we get $S\big(fu,fu,f(gu)\big) = 0$ and therefore $f(gu) = g(gu) = fu = gu$.

On the other hand, if

$$max\left\{\phi\left(S\big(gu,gu,g(gu)\big)\right), \phi\left(\frac{S\big(g(gu),g(gu),f(gu)\big)}{1+S\big(fu,fu,f(gu)\big)}\right)\right\} = \phi\left(\frac{S\big(g(gu),g(gu),f(gu)\big)}{1+S\big(fu,fu,f(gu)\big)}\right)$$

In this case, from (2.21) we have

$$\varphi\left(S\big(fu,fu,f(gu)\big)\right) \le \phi\left(\frac{S\big(g(gu),g(gu),f(gu)\big)}{1+S\big(fu,fu,f(gu)\big)}\right).$$

Now, since $(\varphi,\phi) \in \mathfrak{F}$, we get

$$S\big(fu,fu,f(gu)\big) \le \frac{S\big(g(gu),g(gu),f(gu)\big)}{1+S\big(fu,fu,f(gu)\big)}.$$

Thus, $S\big(fu,fu,f(gu)\big) = 0$ and therefore $f(gu) = g(gu) = fu = gu$.

This result finishes the proof.

By Theorem 2.1, we obtain the following corollaries.

Corollary 2.2: Let $(X,\preceq)$ is a partially ordered set. Suppose that there exist an S-metric S on X such that (X,S) be a complete S-metric space. Let $f,g: X \to X$ be two maps with $f(X) \subset g(X)$, f is g-non-decreasing map and $g(X)$ is closed such that

$$S(fx,fx,fy) \le \alpha S(gx,gx,gy) + \beta\,\frac{S(gy,gy,fy)[1+S(gx,gx,fx)]}{1+S(fx,fx,fy)}, \tag{2.22}$$

for all $x,y \in X$ with $gx \preceq gy$, where $\alpha,\beta > 0$ and $\alpha+\beta < 1$. Assume that if $\{gx_n\}$ is non-decreasing sequence in X such that $gx_n \to gu$, then $gx_n \preceq gu \preceq g(gu)$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point. Furthermore, if f and g commute at the coincidence point, then f and g have a common fixed point.

Proof: Since

$$\begin{aligned} S(fx,fx,fy) &\le \alpha S(gx,gx,gy) + \beta\,\frac{S(gy,gy,fy)[1+S(gx,gx,fx)]}{1+S(fx,fx,fy)}, \\ &\le (\alpha+\beta)\max\left\{S(gx,gx,gy), \frac{S(gy,gy,fy)[1+S(gx,gx,fx)]}{1+S(fx,fx,fy)}\right\} \\ &= \max\left\{(\alpha+\beta)S(gx,gx,gy), (\alpha+\beta)\,\frac{S(gy,gy,fy)[1+S(gx,gx,fx)]}{1+S(fx,fx,fy)}\right\} \end{aligned}$$

for all comparable elements $x,y \in X$, where $\alpha+\beta < 1$. This condition is a particular case of the contractive condition appearing in Theorem 2.1 with the pair of functions $(\varphi,\phi) = (1_{[0,\infty)}, (\alpha+\beta)1_{[0,\infty)}) \in \mathfrak{F}$, given by $\varphi = 1_{[0,\infty)}$ and $\phi = (\alpha+\beta)1_{[0,\infty)}$, (see Example 1.20). Furthermore, we relaxed the requirement of the continuity of mapping to prove the results.

Corollary 2.3: Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist an S-metric S on X such that (X, S) be a complete S-metric space. Let $f, g: X \to X$ be two maps with $f(X) \subset g(X)$, f is g-non-decreasing map and $g(X)$ is closed such that there exists a pair of functions $(\phi, \varphi) \in \mathfrak{F}$ satisfying

$$\varphi\big(S(fx, fx, fy)\big) \leq \phi\big(S(gx, gx, gy)\big) \tag{2.23}$$

for all $x, y \in X$ with $gx \preceq gy$. Assume that if $\{gx_n\}$ is non-decreasing sequence in X such that $gx_n \to gu$, then $gx_n \preceq gu \preceq g(gu)$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point. Furthermore, if f and g commute at the coincidence point, then f and g have a common fixed point.

Corollary 2.4: Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist an S-metric S on X such that (X, S) be a complete S-metric space. Let $f, g: X \to X$ be two maps with $f(X) \subset g(X), f$ is g-non-decreasing map and $g(X)$ is closed such that there exists a pair of functions $(\varphi, \phi) \in \mathfrak{F}$ satisfying

$$\varphi\big(S(fx, fx, fy)\big) \leq \phi\left(\frac{S(gy,gy,fy)[1+S(gx,gx,fx)]}{1+S(fx,fx,fy)}\right), \tag{2.24}$$

for all $x, y \in X$ with $gx \preceq gy$. Assume that if $\{gx_n\}$ is non-decreasing sequence in X such that $gx_n \to gu$, then $gx_n \preceq gu \preceq g(gu)$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point. Furthermore, if f and g commute at the coincidence point, then f and g have a common fixed point.

Taking into account Example 1.19, we have the following corollary.

Corollary 2.5: Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist an S-metric S on X such that (X, S) be a complete S-metric space. Let $f, g: X \to X$ be two maps with $f(X) \subset g(X), f$ is g-non-decreasing map and $g(X)$ is closed such that there exists a pair of functions $(\varphi, \phi) \in \mathfrak{F}$ satisfying

$$\varphi\big(S(fx, fx, fy)\big) \leq max\Big\{\varphi\big(S(gx, gx, gy)\big) - \phi\big(S(gx, gx, gy)\big), \varphi\left(\frac{S(gy,gy,fy)[1+S(gx,gx,fx)]}{1+S(fx,fx,fy)}\right) - \phi\left(\frac{S(gy,gy,fy)[1+S(gx,gx,fx)]}{1+S(fx,fx,fy)}\right)\Big\} \tag{2.25}$$

for all $x, y \in X$ with $gx \preceq gy$. Assume that if $\{gx_n\}$ is non-decreasing sequence in X such that $gx_n \to gu$, then $gx_n \preceq gu \preceq g(gu)$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point. Furthermore, if f and g commute at the coincidence point, then f and g have a common fixed point.

Corollary 2.5 has the following consequences.

Corollary 2.6: Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist an S-metric S on X such that (X, S) be a complete S-metric space. Let $f, g: X \to X$ be two maps with $f(X) \subset g(X), f$ is g-non-decreasing map and $g(X)$ is closed such that there exists a pair of functions $(\varphi, \phi) \in \mathfrak{F}$ satisfying

$$\varphi\big(S(fx, fx, fy)\big) \leq \varphi\big(S(gx, gx, gy)\big) - \phi\big(S(gx, gx, gy)\big), \tag{2.26}$$

for all $x, y \in X$ with $gx \preceq gy$. Assume that if $\{gx_n\}$ is non-decreasing sequence in X such that $gx_n \to gu$, then $gx_n \preceq gu \preceq g(gu)$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point. Furthermore, if f and g commute at the coincidence point, then f and g have a common fixed point.

Corollary 2.7: Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist an S-metric S on X such that (X, S) be a complete S-metric space. Let $f, g: X \to X$ be two maps with $f(X) \subset g(X), f$ is g-non-decreasing map and $g(X)$ is closed such that there exists a pair of functions $(\varphi, \phi) \in \mathfrak{F}$ satisfying

$$\varphi\big(S(fx, fx, fy)\big) \leq \varphi\left(\frac{S(gy,gy,fy)[1+S(gx,gx,fx)]}{1+S(fx,fx,fy)}\right) - \phi\left(\frac{S(gy,gy,fy)[1+S(gx,gx,fx)]}{1+S(fx,fx,fy)}\right), \tag{2.27}$$

for all $x, y \in X$ with $gx \preceq gy$. Assume that if $\{gx_n\}$ is non-decreasing sequence in X such that $gx_n \to gu$, then $gx_n \preceq gu \preceq g(gu)$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point. Furthermore, if f and g commute at the coincidence point, then f and g have a common fixed point.

Taking into account Example 1.20, we have the following corollary.

Corollary 2.8: Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist an S-metric S on X such that (X, S) be a complete S-metric space. Let $f, g: X \to X$ be two maps with $f(X) \subset g(X), f$ is g-non-decreasing map and $g(X)$ is closed such that there exists $\alpha \in S$ satisfying

$$S(fx, fx, fy) \leq max\Big\{\alpha\big(S(gx, gx, gy)\big)S(gx, gx, gy), \\ \alpha\left(\frac{S(gy,gy,fy)[1+S(gx,gx,fx)]}{1+S(fx,fx,fy)}\right)\left(\frac{S(gy,gy,fy)[1+S(gx,gx,fx)]}{1+S(fx,fx,fy)}\right)\Big\} \tag{2.28}$$

for all $x, y \in X$ with $gx \preceq gy$. Assume that if $\{gx_n\}$ is non-decreasing sequence in X such that $gx_n \to gu$, then $gx_n \preceq gu \preceq g(gu)$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point. Furthermore, if f and g commute at the coincidence point, then f and g have a common fixed point.

A consequence of Corollary 2.8 is the following corollary.

Corollary 2.9: Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist an S-metric S on X such that (X, S) be a complete S-metric space. Let $f, g: X \to X$ be two maps with $f(X) \subset g(X), f$ is g-non-decreasing map and $g(X)$ is closed such that there exists $\alpha \in S$ satisfying

$$S(fx, fx, fy) \leq \alpha\big(S(gx, gx, gy)\big)S(gx, gx, gy) \tag{2.29}$$

for all $x, y \in X$ with $gx \preceq gy$. Assume that if $\{gx_n\}$ is non-decreasing sequence in X such that $gx_n \to gu$, then $gx_n \preceq gu \preceq g(gu)$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point. Furthermore, if f and g commute at the coincidence point, then f and g have a common fixed point.

Corollary 2.10: Let $(X, \preceq)$ is a partially ordered set. Suppose that there exist an S-metric S on X such that (X, S) be a complete S-metric space. Let $f, g: X \to X$ be two maps with $f(X) \subset g(X), f$ is g-non-decreasing map and $g(X)$ is closed such that there exists $\alpha \in S$ satisfying

$$S(fx, fx, fy) \leq \alpha\left(\frac{S(gy,gy,fy)[1+S(gx,gx,fx)]}{1+S(fx,fx,fy)}\right)\left(\frac{S(gy,gy,fy)[1+S(gx,gx,fx)]}{1+S(fx,fx,fy)}\right) \tag{2.30}$$

for all $x, y \in X$ with $gx \preceq gy$. Assume that if $\{gx_n\}$ is non-decreasing sequence in X such that $gx_n \to gu$, then $gx_n \preceq gu \preceq g(gu)$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point. Furthermore, if f and g commute at the coincidence point, then f and g have a common fixed point.

If we put $f = g$ in Theorem 2.1, we have following corollary.

Corollary 2.11: Let $(X, \preceq)$ is a partially ordered set. Suppose that there exists an S-metric S on X such that (X, S) be a complete S-metric space. Let $f: X \to X$ be a non-decreasing map such that there exists a pair of functions $(\varphi, \phi) \in \mathfrak{F}$ satisfying

$$\varphi\big(S(fx, fx, fy)\big) \leq max\left\{\phi\big(S(x, x, y)\big), \phi\left(\frac{S(y,y,fy)[1+S(x,x,fx)]}{1+S(fx,fx,fy)}\right)\right\}, \tag{2.31}$$

for all $x, y \in X$ with $x \preceq y$. Assume that if $\{x_n\}$ is non-decreasing sequence in X such that $x_n \to u$, then $x_n \preceq u$ for all $n \in \mathbb{N}$. If there exist $x_0 \in X$ such that $x_0 \preceq fx_0$, then f have a fixed point.

In what follows, we prove a sufficient condition for the uniqueness of the fixed point in Corollary 2.11.

Theorem 2.12: Suppose that: (a) hypothesis of Corollary 2.11 hold, (b) for each $x, y \in X$, there exists $z \in X$ that is comparable to x and y. Then f has a unique fixed point.

Proof: As in the proof of Corollary 2.11, we see that f has a fixed point. Now we prove that the uniqueness of the fixe point of f. Let u and v be two fixed points of f.

We consider the following two cases:

Case 1. u is comparable to v. Then $f^n u$ is comparable to $f^n v$ for all $n \in \mathbb{N}$. For all $a \in X$, applying contractive condition (2.31), we have

$$\begin{aligned}\varphi\big(S(u, u, v)\big) &= \varphi\big(S(f^n u, f^n u, f^n v)\big)\\ &\leq max\left\{\phi\big(S(f^{n-1}u, f^{n-1}u, f^{n-1}v)\big), \phi\left(\frac{S\big(f^{n-1}v, f^{n-1}v, f^n v\big)\big[1+S\big(f^{n-1}u, f^{n-1}u, f^n u\big)\big]}{1+S(f^n u, f^n u, f^n v)}\right)\right\}\\ &= max\left\{\phi\big(S(u, u, v)\big), \phi\left(\frac{S(v,v,v)[1+S(u,u,u)]}{1+S(u,u,v)}\right)\right\}\end{aligned} \tag{2.32}$$

Consider

$$max\left\{\phi\big(S(u, u, v)\big), \phi\left(\frac{S(v, v, v)[1 + S(u, u, u)]}{1 + S(u, u, v)}\right)\right\} = \phi\big(S(u, u, v)\big)$$

Then from (2.33), we have $\varphi\big(S(u, u, v)\big) \leq \phi\big(S(u, u, v)\big)$. Since $(\varphi, \phi) \in \mathfrak{F}$, it follows that $S(u, u, v) = 0$ and so $u = v$.

If

$$max\left\{\phi\big(S(u, u, v)\big), \phi\left(\frac{S(v, v, v)[1 + S(u, u, u)]}{1 + S(u, u, v)}\right)\right\} = \phi\left(\frac{S(v, v, v)[1 + S(u, u, u)]}{1 + S(u, u, v)}\right)$$

Then from (2.33), we have

$$\varphi\big(S(u, u, v)\big) \leq \phi\left(\frac{S(v,v,v)[1+S(u,u,u)]}{1+S(u,u,v)}\right).$$

Then since $(\varphi, \phi) \in \mathfrak{F}$, we have $S(u, u, v) \leq 0$ and so $u = v$

Therefore, in both cases we proved that $u = v$.

Case 2. u is not comparable to v. Then there exists $z \in X$ that is comparable to u and v. Now, we can define the sequence $\{z_n\}$ in X as follows: $z_0 = z,\ fz_n = z_{n+1},\ \forall\, n \in \mathbb{N}$. Since f is non-decreasing we have,

$$z_0 \le z_n \le z_{n+1} \text{ and } \lim_{n\to\infty} S(z_n, z_n, z_{n+1}) = 0. \tag{2.33}$$

As $u \le z_n$, putting $x = u$ and $y = z_n$ in the contractive condition (2.31), we get

$$\begin{aligned}\varphi\big(S(u,u,z_{n+1})\big) &= \varphi\big(S(fu,fu,fz_n)\big)\\ &\le max\left\{\phi\big(S(u,u,z_n)\big), \phi\left(\tfrac{S(z_n,z_n,z_{n+1})[1+S(u,u,fu)]}{1+S(fu,fu,fz_n)}\right)\right\}\\ &= max\left\{\phi\big(S(u,u,z_n)\big), \phi\left(\tfrac{S(z_n,z_n,z_{n+1})}{1+S(u,u,z_{n+1})}\right)\right\}\end{aligned} \tag{2.34}$$

Let us denote

$$G = \left\{n \in \mathbb{N} : \varphi\big(S(u,u,z_{n+1})\big) \le \phi\big(S(u,u,z_n)\big)\right\}$$

$$H = \left\{n \in \mathbb{N} : \varphi\big(S(u,u,z_{n+1})\big) \le \phi\left(\tfrac{S(z_n,z_n,z_{n+1})}{1+S(u,u,z_{n+1})}\right)\right\}$$

Now we remark following again.

(1). If $Card\ G = \infty$, then from (2.34), we can find infinitely natural numbers n satisfying inequality

$$\varphi\big(S(u,u,z_{n+1})\big) \le \phi\big(S(u,u,z_n)\big).$$

Since $(\varphi, \phi) \in \mathfrak{F}$, it follows that the sequence $\{S(u,u,z_{n+1})\}$ is non-increasing and it has a limit $l \ge 0$. Since

$$\lim_{n\to\infty} S(u,u,z_{n+1}) = \lim_{n\to\infty} S(u,u,z_n) = l$$

and $(\varphi, \phi) \in \mathfrak{F}$,we obtain $l = 0$.

(2). If $Card\ H = \infty$, then from (2.34), we can find infinitely natural numbers n satisfying inequality

$$\varphi\big(S(u,u,z_{n+1})\big) \le \phi\left(\tfrac{S(z_n,z_n,z_{n+1})}{1+S(u,u,z_{n+1})}\right).$$

Then since $(\varphi, \phi) \in \mathfrak{F}$, we have

$$S(u,u,z_{n+1}) \le \frac{S(z_n,z_n,z_{n+1})}{1+S(u,u,z_{n+1})}$$

Since$\lim_{n\to\infty} S(z_n,z_n,z_{n+1}) = 0$ and $\lim_{n\to\infty} S(u,u,z_{n+1}) = l$,on making $n \to \infty$ we have $l = 0$.

Therefore, in both cases we proved that

$$\lim_{n\to\infty} S(u,u,z_{n+1}) = l = 0.$$

In the same way it can be deduced that

$$\lim_{n\to\infty} S(v,v,z_{n+1}) = 0.$$

Therefore passing to the limit in

$$S(u,u,v) \leq S(u,u,z_{n+1}) + S(u,u,z_{n+1}) + S(v,v,z_{n+1})$$

as $n \to \infty$, we obtain $u = v$. That is, the fixed point is unique.

3 Example

We give an example to demonstrate the validity of the above result.

Example 3.1 Let $X = \{1, 2, 3\}$ and let S be defined as follows.

$S(1,1,1) = S(2,2,2) = S(3,3,3) = 0,$

$S(1,2,3) = S(1,3,2) = S(2,1,3) = S(3,1,2) = 4,$

$S(2,3,1) = S(3,2,1) = S(1,1,2) = S(1,1,3) = S(2,2,1) = S(3,3,1) = 2,$

$S(2,2,3) = S(3,3,2) = 6,$

$S(2,3,2) = S(3,2,2) = S(3,2,3) = S(2,3,3) = 3,$

$S(1,2,1) = S(2,1,1) = S(1,3,1) = S(3,1,1) = S(2,1,2) = S(1,2,2)$

$= S(3,1,3) = S(1,3,3) = 1.$

We have $S(x,y,z) \geq 0$ for all $x,y,z \in X$ and $S(x,y,z) = 0$ if and only if $x = y = z$. By simple calculations, we see that the inequality

$$S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$$

holds for all $x,y,z,a \in X$. Then S is an S-metric on X with the usual.

Consider the function $f,g : X \to X$ given as $fx = gx = 1, \forall\, x \in X$. Define the functions $\varphi, \phi: [0,\infty) \to [0,\infty)$ as follows: for all $t \in [0,\infty)$, $\varphi(t) = \ln\left(\frac{1}{12} + \frac{5t}{12}\right)$ and $\phi(t) = \ln\left(\frac{1}{12} + \frac{3t}{12}\right)$. Then all assumptions of Theorem 2.1 are satisfied. Then Theorem 2.1 is applicable to f and g on S.

4 Conclusion

In this article, we established some common fixed point theorems for g-monotone maps involving rational expression in the framework of S-metric spaces endowed with a partial order using a class of pairs of functions satisfying certain assumptions. The presented theorems extend, generalize and improve many existing results on metric spaces to S-metric spaces in the literature. Our results may be the motivation to other authors for extending and improving these results to be suitable tools for their applications.

Acknowledgements

The authors express deep gratitude to the referee for his/her valuable comments and suggestions.

Competing Interests

The authors declare that they have no competing interests.

References

[1] Gahler VS. 2-metrische Raume und ihre topologische struktur. Math. Nachr. 1963;26(1963/64):115-118.

[2] Dhage BC. A study of some fixed point theorems. Ph.D. thesis, Marathwada, Aurangabad, India; 1984.

[3] Mustafa Z, Sims B. A new approach to generalized metric spaces. J. Nonlinear Convex Anal. 2006;7(2):289-297.

[4] Sedghi S, Rao KPR, Shobe N. Common fixed point theorems for six weakly compatible mappings in D^*-metric spaces. Internat. J. Math. Math. Sci. 2007;6:225-237.

[5] Dass BK, Gupta S. An extension of Banach contraction principle through rational expressions. Indian J. Pure Appl. Math. 1975;6:1455-1458.

[6] Karapinar E, Shatanawi W, Tas K. Fixed point theorems on partial metric spaces involving rational expressions. Miskolc Math. Notes. 2013;14:135-142.

[7] Kutbi MA, Ahmad J, Hussain N, Arshad M. Common fixed point results for mappings with rational expressions. 2013;Article ID 549518:11.

[8] Arshad M, Khan S, Ahmad J. Fixed point results for f-contractions involving some new rational expressions. JP Journal of Fixed Point Theory and Applications. 2016;11(1):79-97.

[9] Cabrera I, Harjani J, Sadarangani K. A fixed point theorem for contractions of rational type in partially ordered metric spaces. Ann. Univ. Ferrara. 2013;59:251-258.

[10] Sedghi S, Shobe N, Aliouche A. A generalization of fixed point theorem in S-metric spaces. Mat. Vesnik. 2012;64:258-266.

[11] Dung NV. On coupled common fixed points for mixed weakly monotone maps in partially ordered S-metric spaces. Fixed Point Theory Appl. 2013;48:1-17.

[12] Dung NV, Hieu NY, Radojevic S. Fixed point theorems for g-monotone maps on partially ordered S-metric spaces. Filomat. 2014;28(9):1885-1898.
DOI: 10.2298/FIL1409885D

[13] Rocha J, Rzepka B, Sadarangani K. Fixed point theorems for contraction of rational type with PPF dependence in Banach spaces. Journal of Function Spaces. 2014;1-8:Article ID 416187.

[14] Agarwal RP, Karapinar E, Roldan-Lopez-de-Hierro AF. Fixed point theorems in quasi-metric spaces and applications. J. Nonlinear Covex Anal; 2014.

[15] Bergiz M, Karapinar E, Roldan A. Discussion on generalized-$(\alpha\psi, \beta\varphi)$-contractive mappings via generalized altering distance function and related fixed point theorems. Abstr. Appl. Anal. 2014; Article ID 259768.

[16] Khan MS, Swaleh M, Sessa S. Fixed point theorems by altering distances between the points. Bull. Austr. Math. Soc. 1984;30:1-9.

[17] Moradi S, Farajzadeh A. On the fixed point of (ψ, φ)-weak and generalized (ψ, φ)-weak contraction mappings. Appl. Math. Lett. 2012;25:1257-1262.

[18] Saluja AS, Khan MS, Jhade PK, Fisher B. Some fixed point theorems for mappings involving rational type expressions in partial metric spaces. Applied Mathematics E-Notes. 2015;15:147-161.

[19] Saluja AS, Rashwan RA, Magarde D, Jhade PK. Some result in ordered metric spaces for rational type expressions. Facta Universitatis, Ser. Math. Inform. 2016;31(1):125-138.

[20] Dhage BC. Generalized metric spaces and topological structure. I. An., Stiint. Univ. 'Al.I. Cuza Iasi, Mat. 2000;46:3-24.

[21] Jleli M, Samet B. Remarks on G-metric spaces and fixed point theorems. Fixed Point Theory Appl. 2012;201.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/17065