Asian Research Journal of Mathematics
1(5): 1-14, 2016; Article no.ARJOM.30523

SCIENCEDOMAIN *international*
www.sciencedomain.org

SCIENCEDOMAIN

Vacuum Energy of the Laplacian on the Spheres

Louis Omenyi[1*]

[1]*Department of Mathematics, Computer Science, Statistics and Informatics, Federal University, Ndufu-Alike, Ikwo, Nigeria.*

Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/ARJOM/2016/30523
Editor(s):
(1) Andrej V. Plotnikov, Department of Applied and Calculus Mathematics and CAD, Odessa State Academy of Civil Engineering and Architecture, Ukraine.
Reviewers:
(1) Emanuel Guariglia, University of Salerno, Italy.
(2) Shivalika Saxena, University of Allahabad, India.
(3) Houqing Zhou, Shaoyang University, China.
(4) Massimiliano Ferrara, Mediterranean University of Reggio Calabria, Italy.
Complete Peer review History: http://www.sciencedomain.org/review-history/17270

Original Research Article

Received: 15th November 2016
Accepted: 12th December 2016
Published: 17th December 2016

Abstract

Let Δ_g be the Laplacian on smooth functions on a compact Riemannian manifold (M, g) and ζ_g the associated spectral zeta function. Some special values of the spectral zeta function and their generalisations such as the spectral height and spectral determinant usually defined in terms of the spectral zeta function to be $\zeta'_g(0)$ and $\exp(\zeta'_g(0))$ respectively, have been computed explicitly, see e.g [1,2] and [3]. Another special value of the spectral zeta function which has been a fundamental issue in quantum field theory is the Vacuum (Casimir) energy. Casimir energy is defined, mathematically, via the spectral zeta function as a function on the set of metrics on the manifold by $\zeta_g(-\frac{1}{2})$, [4,5] and [6]. In this paper, a general technique for computing the Casimir energy of the Laplacian on the unit n-dimensional sphere, S^n, by factoring the spectral zeta function through the Riemann zeta function ζ_R is presented.

Keywords: Laplacian; spectral zeta function; Riemann zeta function; Casimir energy.

2010 MSC: 32C81, 11M06.

**Corresponding author: E-mail: omenyi.louis@funai.edu.ng;*

1 Introduction

The study of Vacuum energy (also known as Casimir energy) is believed to originate from the work of Hendrik B. G. Casimir (1909 - 2000), who in the year 1948 pointed out the existence of a force between a pair of neutral perfectly conducting parallel plates, [7] and [8]. The Casimir energy may be thought-of as the energy difference due to the distortion of the vacuum, [6]. The energy difference gives rise to what is known as the Casimir force. Although Casimir energy is a concept arising in quantum field theory with observable consequences in physics, research is on-going in the modern aspects of spectral geometry, formulating the notion in a purely mathematical framework; you may see again [8] and the numerous literature cited there-in.

In most physics literature, the Casimir energy, popularly denoted by E_{cas} is written as sum over the eigenvalues $\omega_k = \sqrt{\lambda_k}$ of the Laplacian on smooth functions on M, i.e $E_{\text{cas}} = \sum_k \omega_k$ which is the spectral zeta function ζ_g at $s = -\frac{1}{2}$. Because the spectral zeta function depends on the choice of metric g, [9,10,4,11,12], the Casimir energy $\zeta_g(-\frac{1}{2})$ is defined via the spectral zeta function as a function on the set of metrics on the manifold M, see [7]. However, this sum is usually divergent and has to be regularized.

The regularization of this a priori divergent sum has been variously addressed using different methods; see e.g [8,13,14,5,2,15,16,17,9] and mostly recently [3] and [6] among many other literature. For example in [7], Elizalde had to split the sphere into hemispheres and imposing Dirichlet boundary condition on one of the hemispheres and Neumann boundary condition on the other to be able compute Casimir energy on $S^n, n = 1,2,3,4$. The authors in [3] followed similar procedure as in [8] to compute functional determinant corresponding to massive Laplacian in arbitrary dimensional spheres. In this paper, a more general method of computing the Casimir energy $\zeta_g(-\frac{1}{2})$ of the Laplacian Δ_g on the n-sphere, S^n which only employs factoring the spectral zeta function $\zeta_g(s)$ through the Riemann zeta function $\zeta_R(s)$ is introduced. This method has an advantage of computing the Casimir energy of arbitrary dimensional spheres less tediously over the method of taking the full sphere as the union of Dirichlet and Neumann problems on hemispheres. This is simply obtained by computing the finite part, FP, of the spectral zeta function, ζ_{S^n}, of the Laplacian on the spheres given by

$$\zeta_{S^n}(s) = 2\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\frac{(n-1)^{2m}}{m!4^m}\frac{\Gamma(s+m)}{\Gamma(s)}\frac{\Gamma(k+n-1)}{\Gamma(n)\Gamma(k+1)}(k+\frac{n-1}{2})^{-2s-2m+1}.$$

for each n where the finite part function, FP, is defined by

$$\text{FP}[f](s) := \begin{cases} f(s) \text{ if } s \text{ is not a pole} \\ \lim_{\varepsilon\to 0}(f(s+\varepsilon) - \frac{\text{Residue}}{\varepsilon}), \text{if } s \text{ is a pole.} \end{cases}$$

We proceed by giving the notion of the Laplacian and the spectral zeta function of the Laplacian on Riemannian manifolds.

2 Basic Concepts

2.1 The Laplacian on the unit n-sphere

The Laplacian on smooth functions on (M, g) is the operator

$$\Delta_g : C^\infty(M) \to C^\infty(M) \tag{2.1}$$

defined in local coordinates by

$$\Delta_g = -\operatorname{div}(\operatorname{grad}) = -\frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial x^i} (\sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j}) \tag{2.2}$$

where g^{ij} are the components of the dual metric, [18, 19] and [20].

The operator Δ_g extends to a self-adjoint operator on $L^2(M) \supset H^2(M) \to L^2(M)$ with compact resolvent. This implies that there exists an orthonormal basis $\{\psi_k\} \in L^2(M)$ consisting of eigenfunctions such that

$$\Delta_g \psi_k = \lambda_k \psi_k \tag{2.3}$$

where the eigenvalues are listed with multiplicities

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots; \tag{2.4}$$

see for example, ([18,20] and [17]). The Laplacian Δ_g thus, has one-dimensional null space consisting precisely of constant functions.

The n-dimensional sphere of radius r, $r \in \mathbb{R}^+$ is the set of points in $\mathbb{R}^{n+1}$ at a distance r from a given central point; i.e $S^n(r) = \{x \in \mathbb{R}^{n+1} : \| x \| = r\}$, [17,21,22] and [2]. We call S^n a unit n-sphere or simply an n-sphere when the radius $r = 1$ and write the unit n-sphere as the set

$$S^n = \{x \in \mathbb{R}^{n+1} : \| x \| = 1\} \tag{2.5}$$

where $\| \cdot \|$ is the usual norm on $L^2(S^n)$, see e.g. [23].

The n-sphere is an n-dimensional compact manifold in $(n+1)$-space of constant positive sectional curvature, namely $+1$, $n \geq 2$. So, in particular, the 0-sphere, 1-sphere and the 2-sphere are respectively a pair of points on a line segment, a circle on a plane and the ordinary sphere in 3-dimension.

Let $f \in S^n$ be any function on the n-sphere and $\tilde{f}$ be its extension to an open neighbourhood of S^n that is constant along rays from the centre of S^n. We say that $f \in C^2(S^n)$ if $\tilde{f}$ is a C^2 function of that neighbourhood. For such functions (not containing $\{0\}$) on S^n the Laplacian Δ_n equals

$$\Delta_n f = \Delta_g \tilde{f} \tag{2.6}$$

where Δ_g on the right-hand side of (2.6) is the usual Laplacian in $\mathbb{R}^{n+1}$.

In $\mathbb{R}^n, n \geq 2,$ every point $x \neq 0$ can be represented in polar coordinates as a couple (r, θ) where $r := |x| > 0$ is the polar radius and $\theta := \frac{x}{|x|} \in S^{n-1}$ is the polar angle. In the polar coordinates, the Riemannian measure on S^n is given by $dV = \sin^{n-1} r dr d\theta.$

For the unit n-sphere, the Laplacian (2.6) in polar coordinates reduces to

$$\Delta_n = \frac{1}{\sin^{n-1}\theta} \frac{\partial}{\partial\theta} \{\sin^{n-1}\theta \frac{\partial}{\partial\theta}\} + \frac{1}{\sin^2\theta} \Delta_{n-1} \tag{2.7}$$

where Δ_{n-1} is the Laplacian on S^{n-1}.

Following [24, 14, 22] and [23], we give the following definitions.

Definition 2.1 *A function* $f : \mathbb{R}^n \to \mathbb{C}$ *is called homogeneous of degree* k *if it satisfies* $f(tx) = t^k f(x)$ *for all* $x \in \mathbb{R}^n$ *and* $t > 0$ *fixed.*

Definition 2.2 *Let* $\mathcal{P}_k(n)$ *denote the space of homogeneous polynomials of degree* k *in* $(n+1)$ *variables. The space* $\mathcal{H}_k(n) := \{p_k \in \mathcal{P}_k(n) : \Delta_g p_k = 0, p_k \text{ homogeneous}\}$ *is called the space of harmonic homogeneous polynomials.*

Note, if $p_k \in \mathcal{H}_k(n)$ then

$$p_k(x) = |x|^k \cdot p_k(\frac{x}{|x|}); \text{where} \frac{x}{|x|} \in S^n; x \neq 0.$$

Also, if $p_k |_{S^n} = q_k |_{S^n}$; i.e $p_k(x) = q_k(x) \;\; \forall x \in S^n,$ then

$$p_k(x) = |x|^k \;\; p_k(\frac{x}{|x|}) = |x|^k \;\; q_k(\frac{x}{|x|}) = q_k(x) \;\; \forall x \neq 0; \Rightarrow p_k = q_k$$

since they are both polynomials. One may see [2, 25, 5] for more details.

Theorem 2.3 [23]. *The dimension* $d_k(n)$ *of the space of harmonic polynomial* $\mathcal{H}_k$ *is given by the formula*

$$d_k(n) = \binom{k+n}{n} - \binom{k+n-2}{n}. \tag{2.8}$$

We make the following lemma.

Lemma 2.4 *The multiplicities* $d_k(n)$ *of the eigenspace of the spectrum* $\{\lambda_k\}$ *can be expressed as*

$$d_k(n) = \frac{(2k+n-1)(k+n-2)!}{k!(n-1)!} \tag{2.9}$$

where $k \in \mathbb{N}_0$ *and* $n \geq 1$ is the dimension of the manifold S^n.

Proof. It is clear that

$$d_k(n) = \binom{k+n}{n} - \binom{k+n-2}{n} = \frac{(k+n)!}{k!n!} - \frac{(k+n-2)!}{(k-2)!n!}$$

$$= \frac{(k+n-2)!}{n!}[\frac{(k+n)(k+n-1)}{k!} - \frac{1}{(k-2)!}]$$

which simplifies as

$$\frac{(k+n-2)!}{n!(k-2)!}[\frac{(k+n)(k+n-1)}{k(k-1)} - 1]$$
$$= \frac{(k+n-2)!}{k!}\frac{n}{n!}(2k+n-1)$$
$$= \frac{(2k+n-1)(k+n-2)!}{k!(n-1)!}.$$

2.2 Spectral zeta function

Spectral zeta function is best explained through the well-known Riemann zeta function. Recall that the Riemann zeta function ζ_R is the function defined as $\zeta_R : \{s \in \mathbb{C} : \Re(s) > 1\} \to \mathbb{C}$ with

$$\zeta_R(s) = \sum_{k=1}^{\infty} \frac{1}{k^s};\ \Re(s) > 1 \tag{2.10}$$

c.f: [4] and [10]. Notice that since

$$\sum_{k=1}^{\infty} \left| \frac{1}{k^s} \right| = \sum_{k=1}^{\infty} \frac{1}{k^{\Re(s)}}, \tag{2.11}$$

the series on the right-hand-side of (2.11) converges absolutely if and only if $\Re(s) > 1$. The Riemann zeta function defined by (2.10) above is holomorphic in the region indicated. It, however, admits a meromorphic continuation to the whole s-complex plane with only simple pole at $s=1$ and has residue 1; see e.g. [19,21, 16, 4, 10].

A generalization of the Riemann zeta function (2.10) is the Hurwitz zeta function $\zeta_H(s,a)$.

Definition 2.5 *[4,10] Let* $s \in \mathbb{C}$ *and* $0 < a \le 1$. *Then for* $\Re(s) > 1$, *the Hurwitz zeta function is defined by*

$$\zeta_H(s,a) = \sum_{k=0}^{\infty} \frac{1}{(k+a)^s};\ \Re(s) > 1. \tag{2.12}$$

Clearly, $\zeta_H(s,1) = \zeta_R(s)$. Expression for $a = b+1; b \in \mathbb{R}$ follows by observing that

$$\zeta_H(s,1+b) = \sum_{k=0}^{\infty} \frac{1}{(k+1+b)^s} = \zeta_H(s,b) - \frac{1}{b^s}. \tag{2.13}$$

Theorem 2.6 *For* $0 < a \le 1$, *we have*

$$\zeta_H(s,a) = \frac{1}{a^s} + \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(s+m)}{m!\Gamma(s)} a^m \zeta_R(s+m). \tag{2.14}$$

Proof. Note that for $|z| < 1$ the following binomial expansion holds

$$(1-z)^{-s} = \sum_{m=0}^{\infty} \frac{\Gamma(s+m)}{m!\Gamma(s)} z^m.$$

So for $\Re(s) > 1$, we have

$$\begin{aligned}
\zeta_H(s,a) &= \frac{1}{a^s} + \sum_{k=1}^{\infty} \frac{1}{k^s} \frac{1}{(1+\frac{a}{k})^s} \\
&= \frac{1}{a^s} + \sum_{k=1}^{\infty} \frac{1}{k^s} \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(s+m)}{m!\Gamma(s)} (\frac{a}{k})^m \\
&= \frac{1}{a^s} + \sum_{m=0}^{\infty} (-1)^m \frac{\Gamma(s+m)}{m!\Gamma(s)} (a)^m \sum_{k=1}^{\infty} \frac{1}{k^{s+m}}
\end{aligned}$$

which gives the expansion

Another generalisation of the Riemann zeta function is the spectral zeta function, which is the function of interest in this paper. The Laplacian defined in (2.2) acting on smooth functions on the closed and connected Riemannian n-dimensional manifolds is a non-negative operator and has the discrete spectrum $\{\lambda_k\}_{k=1}^{\infty}$ listed with multiplicities. We define

$$\zeta_g(s) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^s}; \quad \Re(s) > \frac{n}{2}; \tag{2.15}$$

see e.g. [21,11].

The integral kernel, $\zeta_g(s,x,y)$, associated with the spectral zeta function is defined via the operator Δ_g^{-s} (see e.g. [9]) given by the following properties, [23] and [6].

1. It is linear on $L^2(M)$ with 1-dimensional null space consisting of constant functions. This ensures that the smallest eigenvalue of Δ_g^{-s} is 0 of multiplicity 1 with corresponding eigenfunction $\frac{1}{\sqrt{V}}$ where V is the volume of M.
2. The image of Δ_g^{-s} is contained in the orthogonal complement of constant functions in $L^2(M)$ i.e.

   $\int_M \Delta_g^{-s} \psi dV_g = 0 \;\; \forall \psi \in L^2(M)$ constant.

3. $\Delta_g^{-s}\psi_k(x) = \lambda_k^{-s}\psi_k(x)$ for all $\psi_k; k > 0$ an orthonormal basis of eigenfunction of Δ_g.

Then for $\Re(s) > \frac{n}{2}$, we see by property (3.) that Δ_g^{-s} is trace class, with trace given by the spectral zeta function, namely

$$\zeta_g(s) = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^s} = \mathrm{Tr}(\Delta_g^{-s}) = \int_M \zeta_g(s,x,x)dV; \quad \Re(s) > \frac{n}{2}. \tag{2.16}$$

Theorem 2.7 [17,25]. *Let $\{\psi_k\}_{k=1}^{\infty}$ be an orthonormal eigenbasis for Δ_g corresponding to the eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ listed with multiplicities. Then the zeta kernel, $\zeta_g(s,x,y)$, also called the* ***point-wise zeta function****, is equal to*

$$\zeta_g(s,x,y) = \sum_{k=1}^{\infty} \frac{\psi_k(x)\overline{\psi}_k(y)}{\lambda_k^s}; \; \Re(s) > \frac{n}{2}. \tag{2.17}$$

From here on, we suppress the subscript g in $\zeta_g(s)$ and Δ_g. We simply write $\zeta(s)$ and Δ for $\zeta_g(s)$ and Δ_g respectively, unless for purpose of emphasis.

3 Casimir Energy of Δ_g on S^n

Using the properties of the Riemann and Hurwitz zeta functions reviewed above, we can now compute the Casimir energy of the Laplacian Δ_g for the round metric on S^n. As mentioned earlier, sometimes zeta regularisation is not sufficient to define the Casimir energy, because ζ_g has a pole at $-1/2$. In this case, we define the Casimir energy as the finite part of ζ_g at $s=-1/2$, where FP is the finite part function given by

$$\mathrm{FP}[f](s) := \begin{cases} f(s) \text{ if } s \text{ is not a pole} \\ \lim_{\varepsilon\to 0}\left(f(s+\varepsilon)-\frac{\text{Residue}}{\varepsilon}\right), \text{ if } s \text{ is a pole.} \end{cases} \tag{3.1}$$

Now note that the zeta function on the n-sphere is specifically given by

$$\zeta_{S^n}(s) = \sum_{k=1}^{\infty}\frac{d_k(n)}{(k(k+n-1))^s}; \quad \Re(s) > \frac{n}{2} \tag{3.2}$$

with $d_k(n)$ defined by equation (2.8), [2,13,12,17,25,23,5].

The following theorems will be used to switch limits and integrals; and sum and integrals.

Theorem 3.1 *[18,19]. (Dominated Convergence Theorem). Let $\Omega \subseteq \mathbb{R}^n$ be open and let $\{\psi_k\}$ be a sequence of integrable functions on Ω. Suppose that $\lim_{k\to\infty}\psi_k(x) = \psi(x)$ μ -almost everywhere. Further suppose that there exists $\omega \geq 0$ with $\int_\Omega \omega(x)\mathrm{d}\mu(x) < \infty$ such that $\psi_k(x) \leq \omega(x)$ $\forall k$. Then $\psi(x) \leq \omega(x)$ μ -almost everywhere and*

$$\lim_{k\to\infty}\int_\Omega \psi_k(x)\mathrm{d}\mu(x) = \int_\Omega \psi(x)\mathrm{d}\mu(x);$$

where $\mathrm{d}\mu(x)$ is the measure form on Ω.

Theorem 3.2 *[18,19]. (Fubini - Tonelli theorem). Let $\{\psi_k\}$ be a sequence of measurable functions. Sum and integral such as $\sum_k \int \psi_k(x)dx$ can be interchanged in either of the following cases:*

$$\psi_k \geq 0, \forall k \in \mathbb{N} \text{ or } \sum_k \int |\psi_k(x)|\, dx < \infty.$$

Thus $\Re(s) > \frac{n}{2}$ one writes $\zeta_{S^n}(s)$ from (3.2) as

$$\zeta_{S^n}(s) = \frac{1}{(n-1)!}\sum_{k=1}^{\infty}\frac{(k+n-2)!(2k+n-1)}{k!(k(k+n-1))^s}$$

and make the substitution

$$w = k + \frac{n-1}{2} \tag{3.3}$$

to have

$$\zeta_{S^n}(s) = \sum_{w=1}^{\infty} \frac{2w\Gamma(w+\frac{n-1}{2})}{\Gamma(n)\Gamma(w+\frac{3-n}{2})(w^2-\frac{(n-1)^2}{4})^s}.$$

By Mellin transform

$$\zeta_{S^n}(s) = \frac{1}{\Gamma(s)}\int_0^{\infty}\sum_{w=1}^{\infty} \frac{2w\Gamma(w+\frac{n-1}{2})}{\Gamma(n)\Gamma(w+\frac{3-n}{2})} e^{-w^2 t} e^{\frac{(n-1)^2}{4}t} t^{s-1}\mathrm{d}t.$$

Re-arranging of integral and sum using the Fubini - Tonelli theorem (3.2) since the exponential of a negative number is bounded gives

$$\zeta_{S^n}(s) = \frac{1}{\Gamma(s)}\sum_{m=0}^{\infty}\sum_{w=1}^{\infty}\frac{(n-1)^{2m}}{m!4^m} \frac{2w\Gamma(w+\frac{n-1}{2})}{\Gamma(n)\Gamma(w+\frac{3-n}{2})}\int_0^{\infty} e^{-w^2 t} t^{s+m-1}\mathrm{d}t.$$

Now let $w^2 t \mapsto \tau$ to have

$$\zeta_{S^n}(s) = \frac{1}{\Gamma(s)}\sum_{m=0}^{\infty}\sum_{w=1}^{\infty}\frac{(n-1)^{2m}}{m!4^m} \frac{2w\Gamma(w+\frac{n-1}{2})}{\Gamma(n)\Gamma(w+\frac{3-n}{2})} w^{-2s-2m}\int_0^{\infty} e^{-\tau} \tau^{s+m-1}\mathrm{d}\tau$$

$$= 2\sum_{m=0}^{\infty}\frac{(n-1)^{2m}}{m!4^m}\frac{\Gamma(s+m)}{\Gamma(s)}\sum_{w=1}^{\infty}\frac{w^{-2s-2m+1}\Gamma(w+\frac{n-1}{2})}{\Gamma(n)\Gamma(w+\frac{3-n}{2})}.$$

Thus,

$$\zeta_{S^n}(s) = 2\sum_{m=0}^{\infty}\frac{(n-1)^{2m}}{m!4^m}\frac{\Gamma(s+m)}{\Gamma(s)}\sum_{k=1}^{\infty}\frac{\Gamma(k+n-1)}{\Gamma(n)\Gamma(k+1)}(k+\frac{n-1}{2})^{-2s-2m+1}. \quad (3.4)$$

Consequently, for $n = 1$, ie the unit circle, we have

$$\zeta_{S^1}(s) = \sum_{k=1}^{\infty}\frac{2}{k^{2s}} = 2\zeta_R(2s) \quad (3.5)$$

which gives

$$\zeta_{S^1}(-\frac{1}{2}) = 2\zeta_R(-1) \approx -0.166667. \tag{3.6}$$

For S^2, we have

$$\zeta_{S^2}(s) = \sum_{k=1}^{\infty} \frac{2k+1}{(k(k+1))^s} \tag{3.7}$$

and from 3.3 making the substitution $w = k + \frac{1}{2}$ yields

$$\zeta_{S^2} = \sum_{w=1}^{\infty} \frac{2w}{(w^2 - \frac{1}{4})^s}$$

which reduces from (3.4) to

$$\zeta_{S^2} = 2\sum_{m=0}^{\infty} \frac{\Gamma(s+m)}{m! 4^m \Gamma(s)} \zeta_H(2s+2m-1, \frac{3}{2}). \tag{3.8}$$

Therefore, the Casimir energy $\zeta_{S^2}(-\frac{1}{2})$ of Δ_g on S^2 becomes

$$\mathrm{FP}[\zeta_{S^2}(-\frac{1}{2})] \approx -0.265096. \tag{3.9}$$

Similarly, on the 3-sphere, we have

$$\zeta_{S^3}(s) = \sum_{k=1}^{\infty} \frac{(k+1)^2}{(k(k+2))^s} \tag{3.10}$$

and let $w = k+1$ so that

$$\zeta_{S^3}(s) = \sum_{w=1}^{\infty} \frac{w^2}{(w^2-1)^s}$$

and thus

$$\zeta_{S^3}(s) = \sum_{m=0}^{\infty} \frac{\Gamma(s+m)}{m! \Gamma(s)} \zeta_H(2s+2m-2, 2) \tag{3.11}$$

which has simple pole at $m = 2$ with residue $-\frac{1}{6}$ for $\zeta_{S^3}(-\frac{1}{2})$. Therefore,

$$\mathrm{FP}[\zeta_{S^3}(-\frac{1}{2})] \approx -0.411503. \tag{3.12}$$

On S^4, we have

$$\zeta_{S^4}(s) = \frac{1}{6}\sum_{k=1}^{\infty}\frac{(k+1)(k+2)(2k+3)}{(k(k+3))^s} \tag{3.13}$$

and let $w = k + \frac{3}{2}$ to get

$$\zeta_{S^4}(s) = \sum_{w=1}^{\infty}\frac{w(w^2 - \frac{1}{4})}{(w^2 - \frac{9}{4})^s}.$$

From (3.4) this reduces to

$$\zeta_{S^4}(s) = \frac{1}{3}\sum_{m=0}^{\infty}\frac{\Gamma(s+m)}{m!\Gamma(s)}(\frac{9}{4})^m(\zeta_H(2s+2m-3,\frac{5}{2}) - \frac{1}{4}\zeta_H(2s+2m-1,\frac{5}{2})). \tag{3.14}$$

Unsurprisingly, $s = -\frac{1}{2}$ is not a pole for $\zeta_{S^4}(s)$. This gives the Casimir energy of the Laplacian on S^4 as

$$\zeta_{S^4}(-\frac{1}{2}) = \frac{1}{3}\sum_{m=0}^{\infty}\frac{\Gamma(m-1/2)}{m!\Gamma(-1/2)}(\frac{9}{4})^m(\zeta_H(2m-4,\frac{5}{2}) - \frac{1}{4}\zeta_H(2m-2,\frac{5}{2}))$$

$$\approx -0.431743 \tag{3.15}$$

Furthermore, for S^5, we have

$$\zeta_{S^5}(s) = \frac{1}{12}\sum_{k=1}^{\infty}\frac{(k+1)(k+2)^2(k+3)}{(k(k+4))^s}. \tag{3.16}$$

Let $w = k + 2$ to get

$$\zeta_{S^5}(s) = \frac{1}{12}\sum_{w=1}^{\infty}\frac{w^2(w^2-1)}{(w^2-4)^s}.$$

Again from (3.4) this reduces to

$$\zeta_{S^5}(s) = \frac{1}{12}\sum_{m=0}^{\infty}\frac{4^m\Gamma(s+m)}{m!\Gamma(s)}(\zeta_H(2s+2m-4,3) - \zeta_H(2s+2m-2,3)). \tag{3.17}$$

The formula (3.17) has simple poles for $s=-\frac{1}{2}$ at $m=2$ and $m=3$ with residues $\frac{1}{12}$ and $-\frac{1}{6}$ respectively. Hence, the Casimir energy of the Laplacian on S^5 is

$$FP[\zeta_{S^5}(-\frac{1}{2})] = -0.510570. \quad (3.18)$$

The Casimir energy $\zeta_{\Delta_g}(-\frac{1}{2})$ of the Laplacian Δ_g on higher dimensional unit spheres may be computed similarly using the formula (3.4).

4 Conclusion

We have shown that for a given dimension n of the unit sphere, our formula (3.4) reduces to a simpler formula in terms of the Riemann zeta function or its generalisation, the Hurwitz zeta function. The Casimir energy is then simply read-off as

$$FP[\zeta_{S^n}(s)]|_{s=\frac{1}{2}} = FP[2\sum_{m=0}^{\infty}\frac{(n-1)^{2m}}{m!4^m}\frac{\Gamma(s+m)}{\Gamma(s)}\sum_{k=1}^{\infty}\frac{\Gamma(k+n-1)}{\Gamma(n)\Gamma(k+1)}(k+\frac{n-1}{2})^{-2s-2m+1}]|_{s=\frac{1}{2}}.$$

This approach may be employed to compute many other special values of the spectral zeta function of the Laplacian on some other Riemannian manifolds.

Competing Interests

Author has declared that no competing interests exist.

References

[1] Sarnark P. Determinants of Laplacians; heights and finiteness. Analysis ET CETERA, Academic Press, Inc; 1990. ISBN: 0-12-574249-5.

[2] Choi J. Determinants of the Laplacians on the n-dimensional unit sphere. J. Adv. Differ Equations. 2013;236.

[3] Cognola G, Elizalde E, Zerbini S. Functional determinant of the massive Laplace operator and the multiplicative anomaly. Journal of Physics A: Mathematical and Theoretical. 2015;48(4).

[4] Ray DB, Singer IM. *R*-Torsion and the Laplacian on Riemannian manifolds. Adv. Maths. 1971;7: 145-210.

[5] Omenyi L. On the second variation of the spectral zeta function of the Laplacain on homogeneous Riemannian manifolds. British Library; 2014. ISSNI: 0000 0004 5357 4948.
Available:http://ethos.bl.uk/OrderDetails,do?uin=uk.bl.ethos.631609

[6] Omenyi L. Conformal variations of the spectral zeta function of the Laplacian. Journal of Mathematical and Computational Science. 2016;6(6):1024-1046. ISSN: 1927-5307.
Available:http://scik.org

[7] Elizalde E. Ten physical applications of spectral zeta functions. Springer-Verlag, Second Edition; 2012.

[8] Elizalde E, Odintsov SD, et al. Zeta Regularization Techniques with Application. World Scientific, Singapore; 1994.

[9] Seeley R. Complex powers of an elliptic operator: Singular integrals. Am. Math. Soc; Providence. 1967;288-307.

[10] Titchmarsh EC. The Theory of the Riemann Zeta-Function; 2^{nd} edition, revised by D. P. Heath-Brown. The Clarendon Press, Oxford University, New York; 1988.

[11] Osgood B, Philips R, Sarnak, P. Extremals of determinants of Laplacians. J. Functional Analysis. 1988;80:148-211.

[12] Ken R. Critical points of the determinant of the Laplace operator. Journal of Functional Analysis. 1994;122:52-83.

[13] Dowker JS. Massive sphere determinant. J. Phys. A46,285202. arXiv:1404.0986.

[14] Spreafico M. On the homogeneous quadratic Bessel zeta function. Mathematika. 2004;51:123-130.

[15] Choi J, Srivastava HM. An application of the theory of double gamma function. J. Math. 1999;53:209 -222. DMS-797-IR.

[16] Flachi A, Fucci G. Zeta Determinant for Laplace operators on Riemannian cap. Journal of Mathematical Physics; 2010.
DIO: 10.1063/1.3545705

[17] Minakshisundaram S. Zeta functions on the sphere. J. Indian Math. Society. 1949;8:242-256.

[18] Chavel I. Eigenvalues in Riemannian Geometry. Academic Press Inc, London; 1984.

[19] Flajolet P, Gourdon X, Dumas P. Mellin transforms and asymptotics: Harmonic sums. Theoretical Computer Science. 1995;144:3-58.

[20] Zelditch S. On the Generic Spectrum of a Riemannnian Cover. Annales de l'institut Fourier, Tome. 1990;40 (2):407-442.

[21] Minakschisundaram S, Pleijel A. Some Properties of the Eigenfunctions of the Laplace Operator on Riemannian Manifolds. Can. J. Maths. 1949;242-256.

[22] Vardi I. Determinant of Laplacians and multiple gamma functions. SIAM MATH.ANAL. 1988;19(2): 493-507.

[23] Wogu MW. Weyl transforms, heat kernels, green functions and Riemann zeta functions on compact Lie groups. Modern Trends in Pseudo-differential operators. Operator Theory, Advances and Applications, Birkh $\ddot{a}$ user Verlag, Basel/Switzerland. 2006;172:67-85.

[24] Voros A. Zeta functions over zeros of zeta functions. Springer Heidelberg Dorrecht, London, New York; 2010.

[25] Nagase M. Expressions of the heat kernel on spheres by elementary functions and their recurrence relations. Saitama Math. J. 2010;27:25-35.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/17270