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A Note on Banach Contraction Mapping principle in Cone Hexagonal Metric Space

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Authors' contributions

This work was carried out in collaboration between both authors. Author AA proposed the main idea of this paper, performed all the steps of proofs and wrote the first draft of the manuscript. Author EH managed the analysis of the research work and literature searches. Both authors read and approved the final manuscript.

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Abstract

In this paper, we prove fixed point theorem of a self mapping in non-normal cone hexagonal metric spaces. Our result extend and improve some recent results of Azam et al., [Banach contraction principle on cone rectangular metric spaces, Applicable Analysis and Discrete Mathematics, 3 (2), 236 - 241, 2009], Rashwan and Saleh [Some Fixed Point Theorems in Cone Rectangular Metric Spaces, Mathematica Aeterna, 2 (6): 573 - 587, 2012], Garg and Agarwal, [Banach Contraction Principle on Cone Pentagonal Metric Space, J. Adv. Studies Topol., 3 (1), 12 - 18, 2012], Garg, [Banach Contraction Principle on Cone Hexagonal Metric Space, Ultra Scientist, 26 (1), 97 - 103, 2014], and others.

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1 Introduction

In 1922, Banach [1] introduced the concept of Banach contraction mapping principle. Due to wide applications of this concept, the study of existence and uniqueness of fixed points of a mapping and common fixed points of one, two or more mappings has become a subject of great interest. Many authors proved the Banach contraction principle in various generalized metric spaces.

In 2007, Huang and Zhang [2] introduced the concept of a cone metric space, they replaced the set of real numbers by an ordered Banach space and proved some fixed point theorems for contractive type conditions in cone metric spaces. Later on many authors have proved some fixed point theorems for different contractive types conditions in cone metric spaces (for e.g., [3, 4, 5]).

Azam et al. [6] introduced the notion of cone rectangular metric space and proved Banach contraction mapping principle in a normal cone rectangular metric space setting. In 2012, Rashwan and Saleh [7] extended and improved the result of Azam et al. [6] by omitting the assumption of normality condition.

Recently, Garg and Agarwal [8] introduced the notion of cone pentagonal metric space and proved Banach contraction mapping principle in a normal cone pentagonal metric space setting.

Very recently, Garg and Agarwal [9] introduced the notion of cone hexagonal metric space and proved Banach contraction mapping principle in a normal cone hexagonal metric space setting.

In [10], Khamsi claims that most of the cone fixed point results are merely copies of the classical ones and that any extension of known fixed point results to cone metric spaces is redundant; also that underlying Banach space and the associated cone subset are not necessary. In fact, Khamsi's approach includes a small class of results and is very limited since it requires only normal cone metric spaces, so that all results with non-normal cones (which are proper extensions of the corresponding results for metric spaces) cannot be dealt with by his approach (e.g., see [4] and references therein).

Motivated and inspired by the results of [7, 9, 11], it is our purpose in this paper to continue the study of common fixed point of mapping in non-normal cone hexagonal metric space setting. Our results extend and improve the results of [6, 7, 8, 9, 11], and many others.

2 Preliminaries

We present some definitions and Lemmas, which will be needed in the sequel.

Definition 2.1. [2] Let E be a real Banach space and P subset of E. P is called a cone if and only if:

(1) P is closed, nonempty, and $P \neq \{0\}$;

(2) $a, b \in \mathbb{R}, \quad a, b \geq 0$ and $x, y \in P \Longrightarrow ax + by \in P$;

(3) $x \in P$ and $-x \in P \Longrightarrow x = 0$.

Given a cone $P \subseteq E$, we defined a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of P.

Definition 2.2. [2] A cone P is called normal if there is a number $k > 0$ such that for all $x, y \in E$, the inequality

$$0 \leq x \leq y \Longrightarrow \|x\| \leq k\|y\|. \tag{2.1}$$

The least positive number k satisfying (2.1) is called the normal constant of P.

In this paper, we always suppose that E is a real Banach space and P is a cone in E with $int(P) \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 2.3. [2] Let X be a nonempty set. Suppose the mapping $\rho : X \times X \to E$ satisfies:

(1) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

Then ρ is called a cone metric on X, and (X, ρ) is called a cone metric space.

Remark 2.1. The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, \infty)$ (e.g., see [2]).

Definition 2.4. [6] Let X be a nonempty set. Suppose the mapping $\rho : X \times X \to E$ satisfies:

(1) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(3) $\rho(x, y) \leq \rho(x, w) + \rho(w, z) + \rho(z, y)$ for all $x, y, z \in X$ and for all distinct points $w, z \in X - \{x, y\}$ [Rectangular property].

Then ρ is called a cone rectangular metric on X, and (X, ρ) is called a cone rectangular metric space.

Remark 2.2. Every cone metric space is cone rectangular metric space. The converse is not necessarily true (e.g., see [6]).

Definition 2.5. [8] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies:

(1) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for $x, y \in X$;

(3) $d(x, y) \leq d(x, z) + d(z, w) + d(w, u) + d(u, y)$ for all $x, y, z, w, u \in X$ and for all distinct points $z, w, u \in X - \{x, y\}$ [Pentagonal property].

Then d is called a cone pentagonal metric on X, and (X, d) is called a cone pentagonal metric space.

Remark 2.3. Every cone rectangular metric space and so cone metric space is cone pentagonal metric space. The converse is not necessarily true (e.g., see [8]).

Definition 2.6. [9] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies:

(1) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for $x, y \in X$;

(3) $d(x, y) \leq d(x, z) + d(z, w) + d(w, u) + d(u, v) + d(v, y)$ for all $x, y, z, w, u, v \in X$ and for all distinct points $z, w, u, v \in X - \{x, y\}$ [Hexagonal property].

Then d is called a cone hexagonal metric on X, and (X, d) is called a cone hexagonal metric space.

Remark 2.4. Every cone pentagonal metric space and so cone rectangular metric space is cone hexagonal metric space. The converse is not true (e.g., see [9]).

Definition 2.7. [9] Let (X, d) be a cone hexagonal metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ and that for all $n > n_0$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

Definition 2.8. [9] Let (X, d) be a cone hexagonal metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$, with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called Cauchy sequence in X.

Definition 2.9. [9] Let (X, d) be a cone hexagonal metric space. If every Cauchy sequence is convergent in (X, d), then X is called a complete cone hexagonal metric space.

Definition 2.10. [7] Let P be a cone defined as above and let Φ be the set of non decreasing continuous functions $\varphi : P \to P$ satisfying:

1. $0 < \varphi(t) < t$ for all $t \in P \setminus \{0\}$,
2. the series $\sum_{n\geq 0} \varphi^n(t)$ converge for all $t \in P \setminus \{0\}$.

From (1), we have $\varphi(0) = 0$, and from (2), we have $\lim_{n\to 0} \varphi^n(t) = 0$ for all $t \in P \setminus \{0\}$.

Lemma 2.1. [12] *Let (X, d) be a complete cone hexagonal metric space. Let $\{x_n\}$ be a Cauchy sequence in X and suppose that there is natural number N such that:*

1. *$x_n \neq x_m$ for all $n, m > N$;*
2. *x_n, x are distinct points in X for all $n > N$;*
3. *x_n, y are distinct points in X for all $n > N$;*
4. *$x_n \to x$ and $x_n \to y$ as $n \to \infty$.*

Then $x = y$.

3 Main Results

In this section, we derive the main result of our work, which is an extension of Banach contraction principle in cone hexagonal metric space and we give an example to illustrate the result.

Theorem 3.1. *Let (X, d) be a complete cone hexagonal metric space. Suppose the mapping $S : X \to X$ satisfy the following:*

$$d(Sx, Sy) \leq \varphi\big(d(x, y)\big), \tag{3.1}$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then S has a unique fixed point in X.

Proof. Let x_0 be an arbitrary point in X. Define a sequence $\{x_n\}$ in X such that

$$x_{n+1} = Sx_n, \text{ for all } n = 0, 1, 2, \ldots.$$

We assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. Then, from (3.1), it follows that

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Sx_{n-1}, Sx_n) \\ &\leq \varphi\big(d(x_{n-1}, x_n)\big) = \varphi\big(d(Sx_{n-2}, Sx_{n-1})\big) \\ &\leq \varphi^2\big(d(x_{n-2}, x_{n-1})\big) \\ &\;\vdots \\ &\leq \varphi^n\big(d(x_0, x_1)\big). \end{aligned} \tag{3.2}$$

It again follows that

$$\begin{aligned} d(x_n, x_{n+2}) &= d(Sx_{n-1}, Sx_{n+1}) \\ &\leq \varphi\big(d(x_{n-1}, x_{n+1})\big) = \varphi\big(d(Sx_{n-2}, Sx_n)\big) \\ &\leq \varphi^2\big(d(x_{n-2}, x_n)\big) \\ &\ \vdots \\ &\leq \varphi^n\big(d(x_0, x_2)\big). \end{aligned} \tag{3.3}$$

It further follows that

$$\begin{aligned} d(x_n, x_{n+3}) &= d(Sx_{n-1}, Sx_{n+2}) \\ &\leq \varphi\big(d(x_{n-1}, x_{n+2})\big) = \varphi\big(d(Sx_{n-2}, Sx_{n+1})\big) \\ &\leq \varphi^2\big(d(x_{n-2}, x_{n+1})\big) \\ &\ \vdots \\ &\leq \varphi^n\big(d(x_0, x_3)\big), \ and \end{aligned} \tag{3.4}$$

$$\begin{aligned} d(x_n, x_{n+4}) &= d(Sx_{n-1}, Sx_{n+3}) \\ &\leq \varphi\big(d(x_{n-1}, x_{n+3})\big) = \varphi\big(d(Sx_{n-2}, Sx_{n+2})\big) \\ &\leq \varphi^2\big(d(x_{n-2}, x_{n+2})\big) \\ &\ \vdots \\ &\leq \varphi^n\big(d(x_0, x_4)\big). \end{aligned} \tag{3.5}$$

In similar way, for $k = 1, 2, 3, \ldots$, we get

$$d(x_n, x_{n+4k+1}) \leq \varphi^n\big(d(x_0, x_{4k+1})\big), \tag{3.6}$$

$$d(x_n, x_{n+4k+2}) \leq \varphi^n\big(d(x_0, x_{4k+2})\big), \tag{3.7}$$

$$d(x_n, x_{n+4k+3}) \leq \varphi^n\big(d(x_0, x_{4k+3})\big), \tag{3.8}$$

$$d(x_n, x_{n+4k+4}) \leq \varphi^n\big(d(x_0, x_{4k+4})\big). \tag{3.9}$$

By using (3.2) and hexagonal property, we have

$$\begin{aligned} d(x_0, x_5) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) + d(x_4, x_5) \\ &\leq d(x_0, x_1) + \varphi\big(d(x_0, x_1)\big) + \varphi^2\big(d(x_0, x_1)\big) + \varphi^3\big(d(x_0, x_1)\big) + \varphi^4\big(d(x_0, x_1)\big) \\ &\leq \sum_{i=0}^{4} \varphi^i\big(d(x_0, x_1)\big). \end{aligned}$$

Similarly,

$$\begin{aligned} d(x_0, x_9) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) + d(x_4, x_5) \\ &\quad + d(x_5, x_6) + d(x_6, x_7) + d(x_7, x_8) + d(x_8, x_9) \\ &\leq d(x_0, x_1) + \varphi\big(d(x_0, x_1)\big) + \varphi^2\big(d(x_0, x_1)\big) + \varphi^3\big(d(x_0, x_1)\big) + \varphi^4\big(d(x_0, x_1)\big) \\ &\quad + \varphi^5\big(d(x_0, x_1)\big) + \varphi^6\big(d(x_0, x_1)\big) + \varphi^7\big(d(x_0, x_1)\big) + \varphi^8\big(d(x_0, x_1)\big) \\ &\leq \sum_{i=0}^{8} \varphi^i\big(d(x_0, x_1)\big). \end{aligned}$$

Now by induction, we obtain for each $k = 1, 2, 3, \dots$

$$d(x_0, x_{4k+1}) \leq \sum_{i=0}^{4k} \varphi^i\big(d(x_0, x_1)\big). \tag{3.10}$$

Also by (3.2), (3.3) and hexagonal property, we have

$$\begin{aligned} d(x_0, x_6) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) + d(x_4, x_6) \\ &\leq d(x_0, x_1) + \varphi\big(d(x_0, x_1)\big) + \varphi^2\big(d(x_0, x_1)\big) + \varphi^3\big(d(x_0, x_1)\big) + \varphi^4\big(d(x_0, x_2)\big) \\ &\leq \sum_{i=0}^{3} \varphi^i\big(d(x_0, x_1)\big) + \varphi^4\big(d(x_0, x_2)\big). \end{aligned}$$

Similarly,

$$\begin{aligned} d(x_0, x_{10}) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) + d(x_4, x_5) \\ &\quad + d(x_5, x_6) + d(x_6, x_7) + d(x_7, x_8) + d(x_8, x_{10}) \\ &\leq d(x_0, x_1) + \varphi\big(d(x_0, x_1)\big) + \varphi^2\big(d(x_0, x_1)\big) + \varphi^3\big(d(x_0, x_1)\big) + \varphi^4\big(d(x_0, x_1)\big) \\ &\quad + \varphi^5\big(d(x_0, x_1)\big) + \varphi^6\big(d(x_0, x_1)\big) + \varphi^7\big(d(x_0, x_1)\big) + \varphi^8\big(d(x_0, x_2)\big) \\ &\leq \sum_{i=0}^{7} \varphi^i\big(d(x_0, x_1)\big) + \varphi^8\big(d(x_0, x_2)\big). \end{aligned}$$

By induction, we obtain for each $k = 1, 2, 3, \dots$

$$d(x_0, x_{4k+2}) \leq \sum_{i=0}^{4k-1} \varphi^i\big(d(x_0, x_1)\big) + \varphi^{4k}\big(d(x_0, x_2)\big). \tag{3.11}$$

Again by (3.2), (3.4) and hexagonal property, we have

$$\begin{aligned} d(x_0, x_7) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) + d(x_4, x_7) \\ &\leq d(x_0, x_1) + \varphi\big(d(x_0, x_1)\big) + \varphi^2\big(d(x_0, x_1)\big) + \varphi^3\big(d(x_0, x_1)\big) + \varphi^4\big(d(x_0, x_3)\big) \\ &\leq \sum_{i=0}^{3} \varphi^i\big(d(x_0, x_1)\big) + \varphi^4\big(d(x_0, x_3)\big). \end{aligned}$$

Similarly,

$$\begin{aligned} d(x_0, x_{11}) &\leq d(x_0, x_1) + d(x_1, x_2) + d(x_2, x_3) + d(x_3, x_4) + d(x_4, x_5) \\ &\quad + d(x_5, x_6) + d(x_6, x_7) + d(x_7, x_8) + d(x_8, x_{11}) \\ &\leq d(x_0, x_1) + \varphi\big(d(x_0, x_1)\big) + \varphi^2\big(d(x_0, x_1)\big) + \varphi^3\big(d(x_0, x_1)\big) + \varphi^4\big(d(x_0, x_1)\big) \\ &\quad + \varphi^5\big(d(x_0, x_1)\big) + \varphi^6\big(d(x_0, x_1)\big) + \varphi^7\big(d(x_0, x_1)\big) + \varphi^8\big(d(x_0, x_3)\big) \\ &\leq \sum_{i=0}^{7} \varphi^i\big(d(x_0, x_1)\big) + \varphi^8\big(d(x_0, x_3)\big). \end{aligned}$$

So by induction, we obtain for each $k = 1, 2, 3, \dots$

$$d(x_0, x_{4k+3}) \leq \sum_{i=0}^{4k-1} \varphi^i\big(d(x_0, x_1)\big) + \varphi^{4k}\big(d(x_0, x_3)\big). \tag{3.12}$$

In fact, by (3.2), (3.5) and hexagonal property, we have

$$\begin{aligned} d(x_0,x_8) &\le d(x_0,x_1)+d(x_1,x_2)+d(x_2,x_3)+d(x_3,x_4)+d(x_4,x_8) \\ &\le d(x_0,x_1)+\varphi\big(d(x_0,x_1)\big)+\varphi^2\big(d(x_0,x_1)\big)+\varphi^3\big(d(x_0,x_1)\big)+\varphi^4\big(d(x_0,x_4)\big) \\ &\le \sum_{i=0}^{3}\varphi^i\big(d(x_0,x_1)\big)+\varphi^4\big(d(x_0,x_4)\big). \end{aligned}$$

Similarly,

$$\begin{aligned} d(x_0,x_{12}) &\le d(x_0,x_1)+d(x_1,x_2)+d(x_2,x_3)+d(x_3,x_4)+d(x_4,x_5) \\ &\quad + d(x_5,x_6)+d(x_6,x_7)+d(x_7,x_8)+d(x_8,x_{12}) \\ &\le d(x_0,x_1)+\varphi\big(d(x_0,x_1)\big)+\varphi^2\big(d(x_0,x_1)\big)+\varphi^3\big(d(x_0,x_1)\big)+\varphi^4\big(d(x_0,x_1)\big) \\ &\quad +\varphi^5\big(d(x_0,x_1)\big)+\varphi^6\big(d(x_0,x_1)\big)+\varphi^7\big(d(x_0,x_1)\big)+\varphi^8\big(d(x_0,x_4)\big) \\ &\le \sum_{i=0}^{7}\varphi^i\big(d(x_0,x_1)\big)+\varphi^8\big(d(x_0,x_4)\big). \end{aligned}$$

By induction, we obtain for each $k = 1, 2, 3, \ldots$

$$d(x_0,x_{4k+4}) \le \sum_{i=0}^{4k-1}\varphi^i\big(d(x_0,x_1)\big)+\varphi^{4k}\big(d(x_0,x_4)\big). \tag{3.13}$$

Using inequality (3.6) and (3.10) for $k = 1, 2, 3, \ldots$, we have

$$\begin{aligned} d(x_n,x_{n+4k+1}) &\le \varphi^n\big(d(x_0,x_{4k+1})\big) \\ &\le \varphi^n\sum_{i=0}^{4k}\varphi^i\big(d(x_0,x_1)\big) \\ &\le \varphi^n\Big[\sum_{i=0}^{4k}\varphi^i\big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)+d(x_0,x_4)\big)\Big] \\ &\le \varphi^n\Big[\sum_{i=0}^{\infty}\varphi^i\big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)+d(x_0,x_4)\big)\Big]. \end{aligned} \tag{3.14}$$

Similarly for $k = 1, 2, 3, \ldots$, inequalities (3.7) and (3.11) implies that

$$\begin{aligned} d(x_n,x_{n+4k+2}) &\le \varphi^n\big(d(x_0,x_{4k+2})\big) \\ &\le \varphi^n\Big[\sum_{i=0}^{4k-1}\varphi^i\big(d(x_0,x_1)\big)+\varphi^{4k}\big(d(x_0,x_2)\big)\Big] \\ &\le \varphi^n\Big[\sum_{i=0}^{4k-1}\varphi^i\big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)+d(x_0,x_4)\big) \\ &\qquad +\varphi^{4k}\big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)+d(x_0,x_4)\big)\Big] \\ &\le \varphi^n\Big[\sum_{i=0}^{4k}\varphi^i\big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)+d(x_0,x_4)\big)\Big] \\ &\le \varphi^n\Big[\sum_{i=0}^{\infty}\varphi^i\big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)+d(x_0,x_4)\big)\Big]. \end{aligned} \tag{3.15}$$

Again for $k = 1, 2, 3, \ldots$, inequalities (3.8) and (3.12) implies that

$$\begin{aligned} d(x_n, x_{n+4k+3}) &\leq \varphi^n\big(d(x_0, x_{4k+3})\big) \\ &\leq \varphi^n\Big[\sum_{i=0}^{4k-1}\varphi^i\big(d(x_0,x_1)\big) + \varphi^{4k}\big(d(x_0,x_3)\big)\Big] \\ &\leq \varphi^n\Big[\sum_{i=0}^{4k-1}\varphi^i\big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)+d(x_0,x_4)\big) \\ &\qquad + \varphi^{4k}\big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)+d(x_0,x_4)\big)\Big] \\ &\leq \varphi^n\Big[\sum_{i=0}^{4k}\varphi^i\big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)+d(x_0,x_4)\big)\Big] \\ &\leq \varphi^n\Big[\sum_{i=0}^{\infty}\varphi^i\big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)+d(x_0,x_4)\big)\Big]. \end{aligned} \tag{3.16}$$

Again for $k = 1, 2, 3, \ldots$, inequalities (3.9) and (3.13) implies that

$$\begin{aligned} d(x_n, x_{n+4k+4}) &\leq \varphi^n\big(d(x_0, x_{4k+4})\big) \\ &\leq \varphi^n\Big[\sum_{i=0}^{4k-1}\varphi^i\big(d(x_0,x_1)\big) + \varphi^{4k}\big(d(x_0,x_4)\big)\Big] \\ &\leq \varphi^n\Big[\sum_{i=0}^{4k-1}\varphi^i\big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)+d(x_0,x_4)\big) \\ &\qquad + \varphi^{4k}\big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)+d(x_0,x_4)\big)\Big] \\ &\leq \varphi^n\Big[\sum_{i=0}^{4k}\varphi^i\big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)+d(x_0,x_4)\big)\Big] \\ &\leq \varphi^n\Big[\sum_{i=0}^{\infty}\varphi^i\big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)+d(x_0,x_4)\big)\Big]. \end{aligned} \tag{3.17}$$

Thus, by inequalities (3.14), (3.15), (3.16) and (3.17) we have, for each m,

$$d(x_n, x_{n+m}) \leq \varphi^n\Big[\sum_{i=0}^{\infty}\varphi^i\big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)+d(x_0,x_4)\big)\Big]. \tag{3.18}$$

Since $\sum_{i=0}^{\infty}\varphi^i\big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)+d(x_0,x_4)\big)$ converges (by definition 2.10), where $d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)+d(x_0,x_4) \in P \setminus \{0\}$ and P is closed, then $\sum_{i=0}^{\infty}\varphi^i\big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)+d(x_0,x_4)\big) \in P \setminus \{0\}$. Hence

$$\lim_{n\to\infty}\varphi^n\Big[\sum_{i=0}^{\infty}\varphi^i\big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)+d(x_0,x_4)\big)\Big] = 0.$$

Then for given $c \gg 0$, there is a natural number N_1 such that

$$\varphi^n\Big[\sum_{i=0}^{\infty}\varphi^i\big(d(x_0,x_1)+d(x_0,x_2)+d(x_0,x_3)+d(x_0,x_4)\big)\Big] \ll c, \quad \forall n \geq N_1. \tag{3.19}$$

Thus from (3.18) and (3.19), we have

$$d(x_n, x_{n+m}) \ll c, \text{ for all } n \geq N_1.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in (X, d). Since X is complete, then there exists a point $z \in X$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} Sx_{n-1} = z$.

Now, we will show that z is a fixed point of S, i.e. $Sz = z$. Given $c \gg 0$, we choose $N_2, N_3 \in \mathbb{N}$ such that $d(z, x_n) \ll \frac{c}{5}, \quad \forall n \geq N_2$ and $d(x_n, x_{n+1}) \ll \frac{c}{5}, \quad \forall n \geq N_3$.
Since $x_n \neq x_m$ for $n \neq m$, therefore by hexagonal property, we have

$$\begin{aligned} d(Sz, z) &\leq d(Sz, Sx_n) + d(Sx_n, Sx_{n+1}) + d(Sx_{n+1}, Sx_{n+2}) + d(Sx_{n+2}, Sx_{n+3}) + d(Sx_{n+3}, z) \\ &\leq \varphi\big(d(z, x_n)\big) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, z) \\ &< d(z, x_n) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4}) + d(x_{n+4}, z) \\ &\ll \frac{c}{5} + \frac{c}{5} + \frac{c}{5} + \frac{c}{5} + \frac{c}{5} = c, \text{ for all } n \geq N, \end{aligned}$$

where $N = \max\{N_2, N_3\}$. Since c is arbitrary we have $d(Sz, z) \ll \frac{c}{m}, \quad \forall m \in \mathbb{N}$. Since $\frac{c}{m} \to 0$ as $m \to \infty$, we conclude $\frac{c}{m} - d(Sz, z) \to -d(Sz, z)$ as $m \to \infty$. Since P is closed, $-d(Sz, z) \in P$. Hence $d(Sz, z) \in P \cap -P$. By definition of cone we get that $d(Sz, z) = 0$, and so $Sz = z$. Therefore, S has a fixed point that is z in X.

Next we show that z is unique. For suppose z' be another fixed point of S such that $Sz' = z'$. Therefore,

$$d(z, z') = d(Sz, Sz') \leq \varphi\big(d(z, z')\big) < d(z, z').$$

Hence, $z = z'$. This completes the proof of the theorem. □

The following example illustrates the result of Theorem 3.1.

Example Let $X = \{1, 2, 3, 4, 5, 6\}$, $E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$ is a cone in E. Define $d : X \times X \to E$ as follows:

$$d(x, x) = 0, \ \forall x \in X;$$
$$d(1, 2) = d(2, 1) = (5, 10);$$
$$\begin{aligned} d(1,3) &= d(3,1) = d(1,4) = d(4,1) = d(1,5) = d(5,1) = d(2,3) = d(3,2) = d(2,4) = d(4,2) \\ &= d(2,5) = d(5,2) = d(3,4) = d(4,3) = d(3,5) = d(5,3) = d(4,5) = d(5,4) = (1,2); \end{aligned}$$
$$d(1,6) = d(6,1) = d(2,6) = d(6,2) = d(3,6) = d(6,3) = d(4,6) = d(6,4) = d(5,6) = d(6,5) = (4,8).$$

Then (X, d) is a complete cone hexagonal metric space, but (X, d) is not a complete cone pentagonal metric space because it lacks the pentagonal property:

$$\begin{aligned} (5, 10) = d(1, 2) &> d(1, 3) + d(3, 4) + d(4, 5) + d(5, 2) \\ &= (1, 2) + (1, 2) + (1, 2) + (1, 2) \\ &= (4, 8), \text{ as } (5, 10) - (4, 8) = (1, 2) \in P. \end{aligned}$$

Now, we define a mapping $S : X \to X$ as follows

$$S(x) = \begin{cases} 5, & \text{if } x \neq 6; \\ 2, & \text{if } x = 6. \end{cases}$$

Hence, we obtain that

$$\begin{aligned} d(S(1), S(2)) &= d(S(1), S(3)) = d(S(1), S(4)) = d(S(1), S(5)) = d(S(2), S(3)) \\ &= d(S(2), S(4)) = d(S(2), S(5)) = d(S(3), S(4)) = d(S(3), S(5)) = 0. \end{aligned}$$

And in all other cases $d(S(x), S(y)) = (1, 2)$ and $d(x, y) = (4, 8)$, for all $x, y \in X$.

Thus, the conditions of Theorem 3.1 holds for all $x, y \in X$, where $\varphi(t) = \frac{1}{4}t$, and $5 \in X$ is the unique fixed point of the mappings S.

4 Conclusion

Corollary 4.1. *Let (X, d) be a complete cone hexagonal metric space. Suppose the mapping $S : X \to X$ satisfy the following:*

$$d(S^m x, S^m y) \leq \varphi\big(d(x, y)\big), \tag{4.1}$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then S has a unique fixed point in X.

Proof. From Theorem 3.1, we conclude that S^m has a fixed point say z. Hence

$$Sz = S(S^m z) = S^{m+1} z = S^m(Sz). \tag{4.2}$$

Then Sz is also a fixed point to S^m. By uniqueness of z, we have $Sz = z$. □

Corollary 4.2. (see [11]) *Let (X, d) be a complete cone pentagonal metric space. Suppose the mapping $S : X \to X$ satisfy the following:*

$$d(S^m x, S^m y) \leq \varphi\big(d(x, y)\big),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then S has a unique fixed point in X.

Proof. This follows from the Remark 2.4 and Corollary 4.1. □

Corollary 4.3. (see [7]) *Let (X, d) be a complete cone rectangular metric space. Suppose the mapping $S : X \to X$ satisfy the following:*

$$d(S^m x, S^m y) \leq \varphi\big(d(x, y)\big),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then S has a unique fixed point in X.

Proof. This follows from the Remark 2.3 and Corollary 4.1. □

Corollary 4.4. (see [11]) *Let (X, d) be a complete cone pentagonal metric space. Suppose the mapping $S : X \to X$ satisfy the following:*

$$d(Sx, Sy) \leq \varphi\big(d(x, y)\big),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then S has a unique fixed point in X.

Proof. This follows from the Remark 2.4 and Theorem 3.1. □

Corollary 4.5. (see [7]) *Let (X, d) be a complete cone rectangular metric space. Suppose the mapping $S : X \to X$ satisfy the following:*

$$d(Sx, Sy) \leq \varphi\big(d(x, y)\big),$$

for all $x, y \in X$, where $\varphi \in \Phi$. Then S has a unique fixed point in X.

Proof. This follows from the Remark 2.3 and Theorem 3.1. □

Corollary 4.6. (see [9]) *Let (X, d) be a cone hexagonal metric space, P be a normal cone, and the mapping $S : X \to X$ satisfy the following:*

$$d(Sx, Sy) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then S has a unique fixed point in X.

Proof. Define $\varphi : P \to P$ by $\varphi(t) = \lambda t$. Then it is clear that φ satisfies the conditions in definition 2.10. Hence the results follows from Theorem 3.1. □

Corollary 4.7. (see [8]) *Let (X, d) be a cone pentagonal metric space, P be a normal cone, and the mapping $S : X \to X$ satisfy the following:*

$$d(Sx, Sy) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then S has a unique fixed point in X.

Proof. This follows from the Remark 2.4 and Corollary 4.6. □

Corollary 4.8. (see [6]) *Let (X, d) be a cone rectangular metric space, P be a normal cone, and the mapping $S : X \to X$ satisfy the following:*

$$d(Sx, Sy) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then S has a unique fixed point in X.

Proof. This follows from the Remark 2.3 and Corollary 4.7. □

Corollary 4.9. (see [2]) *Let (X, d) be a cone metric space, P be a normal cone, and the mapping $S : X \to X$ satisfy the following:*

$$d(Sx, Sy) \leq \lambda d(x, y),$$

for all $x, y \in X$, where $\lambda \in [0, 1)$. Then S has a unique fixed point in X.

Proof. This follows from the Remark 2.2 and Corollary 4.8. □

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Competing Interests

The authors declare that no competing interests exist.

References

[1] Banach S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundamenta Mathematicae 1922;3:133-181.

[2] Huang LG, Zhang X. Cone metric spaces and fixed point theorems of contractive mappings. Journal of Mathematical Analysis and Applications 2007;332(2):1468-1476.

[3] Abbas M, Jungck G. Common fixed point results for non commuting mappings without continuity in cone metric spaces. Journal of Mathematical Analysis and Applications 2008;341(1):416-420.

[4] Janković, et al. On cone metric spaces: A survey, Nonlinear Analysis: Theory, Methods & Appl. 2011;74(7):2591-2601.

[5] Rezapour S, Hamlbarani R. Some notes on paper cone metric spaces and fixed point theorems of contractive mappings. Journal of Mathematical Analysis and Applications. 2008;345(2):719-724.

[6] Azam A, et al. Banach contraction principle on cone rectangular metric spaces. Applicable Analysis and Discrete Mathematics. 2009;3:236-241.

[7] Rashwan RA, Saleh SM. Some fixed point theorems in cone rectangular metric spaces. Mathematica Aeterna. 2012;2(6):573-587.

[8] Garg M, Agarwal S. Banach contraction principle on cone pentagonal metric space. Journal of Advanced Studies in Topology. 2012;3(1):12-18.

[9] Garg M. Banach contraction principle on cone hexagonal metric space. Ultra Scientist. 2008;26(1):97-103.

[10] Khamsi MA. Remarks on cone metric spaces and fixed point theorems of contractive mappings. Fixed Point Theory Appl. 2010;7. Article ID 315398
DOI: 10.1115/2010/315398

[11] Auwalu A. Banach fixed point theorem in a Cone pentagonal metric spaces. Journal of Advanced Studies in Topology. 2016;7(2):60-67.

[12] Auwalu A, Hınçal E. The Kannan's fixed point theorem in a cone hexagonal metric spaces. Advances in Research. 2016;7(1):1-9.

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