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New Kinds of Hypergeometric Matrix Functions

Ayman Shehata[1,2*]

[1] *Department of Mathematics, Faculty of Science, Assiut University, Assiut 71516, Egypt.*
[2] *Department of Mathematics, College of Science and Arts in Unaizah, Qassim University, Qassim 10363, Kingdom of Saudi Arabia.*

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Abstract

The present paper deals with the definition and study of new kinds of hypergeometric matrix functions within complex analysis. We also get the radius of regularity and matrix recurrence relations are then developed for $l(m,n)$-hypergeometric matrix function of two complex variables. We give a different approach to prove the effect of the differential operator on this function. Finally, another hypergeometric matrix function, namely, $p\ l(m,n)$-hypergeometric matrix function of two complex variables are defined, its components when the positive integer p is greater than one, provide a matrix partial differential equation satisfied by these function and some of their properties are investigated.

Keywords: $l(m,n)$*-Hypergeometric matrix function;* $p\ l(m,n)$*-Hypergeometric matrix function; Matrix differential equation; Differential operator.*

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**Corresponding author: E-mail: drshehata2006@yahoo.com*

1 Introduction

The study of special matrix polynomials is important due to their applications in various critical areas of statistics [1], engineering, applied mathematics, mathematical physics, theoretical physics, group representation theory and Lie Groups theory [2]. In the present sequel to these recent works (see [3], [4], [5], [6], [7], [8], [9], [10]), extension to the matrix function framework of the classical families of p-Kummer's matrix function, p and q-Appell, hypergeometric matrix function and Humbert matrix function have been proposed. Motivated by the work of the author in the subsequent developments presented in (see [11], [12]), who has earlier studied the p and q-Horn's H_2, $pl(m,n)$-Kummer's matrix functions of two complex variables under differential operators. The reason of interest for this family of hypergeometric function is due to their intrinsic mathematical importance and the fact that these functions have applications in physics.

Our main aim in this paper is to prove new properties for the $l(m,n)$-hypergeometric matrix function within complex analysis. The outline of this paper is as follows: In Section 2, the radius of regularity and matrix recurrence relations on $l(m,n)$-hypergeometric matrix function of two complex variables are established. The effect of differential operator $\alpha(\mathbb{D})$ on this function is investigated. Finally, we introduce $p\, l(m,n)$ hypergeometric matrix function and derive their properties such as the radius of regularity and matrix partial differential equation in Section 3.

1.1 Preliminaries

In this section, we will give some necessary definition, facts, notations, theorem, and mathematical preliminaries which are used further in this paper.

Throughout this paper, if A is a matrix in $\mathbb{C}^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all the eigenvalues of A. I and **O** will denote the identity matrix and the null matrix in $\mathbb{C}^{N\times N}$, respectively. Its two-norm will be denoted by $||A||_2$ and defined by (see [13])

$$||A||_2 = \sup_{x\neq 0} \frac{||Ax||_2}{||x||_2}$$

where for a vector y in $\mathbb{C}^N$, $||y||_2$ denotes the Euclidean norm of y, $||y||_2 = (y^T y)^{\frac{1}{2}}$.

Fact 1.1. *(Dunford and Schwartz [14]) Let $f(z)$ and $g(z)$ be holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane, and if A, B are matrices in $\mathbb{C}^{N\times N}$ such that $\sigma(A) \subset \Omega$, $\sigma(B) \subset \Omega$ and $AB = BA$, then from the properties of the matrix functional calculus, it follows that*

$$f(A)g(B) = g(B)f(A). \tag{1.1}$$

Fact 1.2. *(Jódar and Cortés [15]) Let A be a matrix in $\mathbb{C}^{N\times N}$ such that*

$$A + nI \quad \text{is an invertible matrix} \quad \text{for all integers } n \geq 0 \tag{1.2}$$

Definition 1.1. (Jódar and Cortés [15]) For $A \in \mathbb{C}^{N\times N}$, the Pochhammer symbol (the shifted factorial) is defined as

$$(A)_n = A(A+I)...(A+(n-1)I) = \Gamma(A+nI)/\Gamma(A), \quad n \geq 1, \quad (A)_0 = I, \tag{1.3}$$

where $\Gamma(A)$ is an invertible matrix, its inverse coincides with $\Gamma^{-1}(A) = \frac{1}{\Gamma(A)}$.

Theorem 1.1. *(Jódar and Cortés [16]) Jódar and Cortés have proved :*

$$\Gamma(A) = \lim_{n\to\infty} (n-1)![(A)_n]^{-1} n^A, \tag{1.4}$$

where A is a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition $\Re(z) > 0$, for every eigenvalue $z \in \sigma(A)$.

Fact 1.3. *(Golub and Van Loan [13]) For any arbitrary matrices A and B in $\mathbb{C}^{N\times N}$, then*

$$\begin{aligned} \|AB\| &\le \|A\|\|B\|, \\ \|(A)_n\| &\le (\|A\|)_n, \\ \|(A)_n(B)_n\| &\le (\|A\|)_n(\|B\|)_n. \end{aligned} \tag{1.5}$$

Fact 1.4. *(Jódar and Cortés [17]) If A in $\mathbb{C}^{N\times N}$ and let us denote the real numbers $\alpha(A)$ and $\beta(A)$ such that $\alpha(A) = \max\{\Re(z) : z \in \sigma(A)\}$ and $\beta(A) = \min\{\Re(z) : z \in \sigma(A)\}$. Then*

$$\begin{aligned} \|e^{tA}\| &\le e^{t\alpha(A)} \sum_{k=0}^{r-1} \frac{(\|A\| r^{\frac{1}{2}} t)^k}{k!}; \quad t \ge 0, \\ \|n^A\| &\le n^{\alpha(A)} \sum_{k=0}^{r-1} \frac{(\|A\| r^{\frac{1}{2}} \ln n)^k}{k!}; \quad n \ge 1. \end{aligned} \tag{1.6}$$

Fact 1.5. *If n is large enough, then for C a matrix in $\mathbb{C}^{N\times N}$ such that $C + nI$ is an invertible matrix for all integers $n \ge 0$, then we will mention the following relation which appeared in Jódar and Cortés [15] in the form :*

$$\|(C + nI)^{-1}\| \le \frac{1}{n - \|C\|}, \quad n > \|C\|. \tag{1.7}$$

Notation 1.1. *Let A be an arbitrary matrix in $\mathbb{C}^{N\times N}$. For any natural number $l(m, n) \ge 0$, we can write the following relations*

$$(A + (l(m, n) - 1)I)! \cong \sqrt{2\pi(A + (l(m, n) - 1)I)} \left(\frac{(A + (l(m, n) - 1)I)}{e} \right)^{(A+(l(m,n)-1)I)} \tag{1.8}$$

and

$$\sigma_{m,n} = \begin{cases} (\frac{m+n}{m})^{\frac{m}{2}} (\frac{m+n}{n})^{\frac{n}{2}}, & m,\ n \ne 0, \\ 1, & m,\ n = 0, \end{cases} \tag{1.9}$$

where $l(m, n) = \frac{1}{2}(m + n + 1)(m + n) + n$ in [18].

Definition 1.2. (Sayyed [18]) Let N be a finite positive integer. Then the differential operator $\Xi(\mathbb{D})$ is defined as

$$\Xi(\mathbb{D}) = 1 + \sum_{k=1}^{N} \mathbb{D}^k, \quad \mathbb{D}^k = \mathbb{D}\, \mathbb{D}^{k-1}. \tag{1.10}$$

2 $l(m, n)$-Hypergeometric Matrix Function of Two Complex Variables

Let A, B and C be commutative matrices in $C^{N\times N}$ such that $C + l(m, n)I$ is an invertible matrix for all integers $l(m, n) \ge 0$. We define the $l(m, n)$-hypergeometric matrix function of two complex variables in the form

$${}_2F_1^{l(m,n)}(A, B; C; z, w) = \sum_{l(m,n)\ge 0} \frac{(A)_{l(m,n)}(B)_{l(m,n)}[(C)_{l(m,n)}]^{-1}}{(l(m, n))!} z^m w^n = \sum_{l(m,n)\ge 0} U_{m,n}(z, w) \tag{2.1}$$

where $U_{m,n} = \frac{(A)_{l(m,n)}(B)_{l(m,n)}[(C)_{l(m,n)}]^{-1}}{(l(m,n))!}$, $U_{m,n}(z, w) = \frac{(A)_{l(m,n)}(B)_{l(m,n)}[(C)_{l(m,n)}]^{-1}}{(l(m,n))!} z^m w^n$ and $l(m, n)$ is a non negative integer.

For simplicity, we can write the ${}_2F_1^{l(m,n)}(A\pm I, B; C; z, w)$ in the form ${}_2F_1^{l(m,n)}(A\pm I)$, ${}_2F_1^{l(m,n)}(A, B\pm I; C; z, w)$ in the form ${}_2F_1^{l(m,n)}(B \pm I)$ and ${}_2F_1^{l(m,n)}(A, B; C \pm I; z, w)$ in the form ${}_2F_1^{l(m,n)}(C \pm I)$.

Now, we prove that the study of this function by calculating its radius of convergence R. For $1 \leq \sigma_{m,n} \leq 2^{\frac{m+n}{2}}$ (see [18]), we get

$$\begin{aligned}
\frac{1}{R} &= \lim_{m+n\to\infty} \sup \left(\frac{\|U_{m,n}\|}{\sigma_{m,n}} \right)^{\frac{1}{m+n}} \\
&= \lim_{m+n\to\infty} \sup \left\| \left(\frac{(A)_{l(m,n)}(B)_{l(m,n)}[(C)_{l(m,n)}]^{-1}}{(l(m,n))!\sigma_{m,n}} \right) \right\|^{\frac{1}{m+n}} \\
&= \lim_{m+n\to\infty} \sup \left\| \left(\Gamma^{-1}(A)\Gamma(A + l(m,n)I)\Gamma^{-1}(B)\Gamma(B + l(m,n)I) \right) \right. \\
&\quad \left. \left(\Gamma(C)\Gamma^{-1}(C + l(m,n)I) \frac{1}{(l(m,n))!\sigma_{m,n}} \right) \right\|^{\frac{1}{m+n}} \\
&\leq \lim_{m+n\to\infty} \sup \left\| \left(\frac{(A + (l(m,n) - 1)I)}{e} \right)^{(A+(l(m,n)-1)I)} \left(\frac{(B + (l(m,n) - 1)I)}{e} \right)^{(B+(l(m,n)-1)I)} \right. \\
&\quad \left. \left(\frac{(C + (l(m,n) - 1)I)}{e} \right)^{-(C+(l(m,n)-1)I)} \left(\frac{l(m,n)}{e} \right)^{-l(m,n)} \frac{1}{\sigma_{m,n}} \right\|^{\frac{1}{m+n}} \\
&\leq \lim_{m+n\to\infty} \sup \left\| \left(\frac{(A + (l(m,n) - 1)I)}{e} \right)^{l(m,n)I} \left(\frac{(A + (l(m,n) - 1)I)}{e} \right)^{A-I} \right. \\
&\quad \left(\frac{(B + (l(m,n) - 1)I)}{e} \right)^{l(m,n)I} \left(\frac{(B + (l(m,n) - 1)I)}{e} \right)^{B-I} \\
&\quad \left. \left(\frac{(C + (l(m,n) - 1)I)}{e} \right)^{-l(m,n)I} \left(\frac{(C + (l(m,n) - 1)I)}{e} \right)^{-(C-I)} \left(\frac{l(m,n)}{e} \right)^{-l(m,n)} \frac{1}{\sigma_{m,n}} \right\|^{\frac{1}{m+n}} \\
&\leq \lim_{m+n\to\infty} \sup \left\| \left(I + \frac{A-I}{l(m,n)} \right)^{l(m,n)I} \left(I + \frac{A-I}{l(m,n)} \right)^{A-I} \left(I + \frac{B-I}{l(m,n)} \right)^{l(m,n)I} \left(I + \frac{B-I}{l(m,n)} \right)^{B-I} \right. \\
&\quad \left. \left(I + \frac{C-I}{l(m,n)} \right)^{-l(m,n)I} \left(I + \frac{C-I}{l(m,n)} \right)^{-(C-I)} \left(\frac{e}{l(m,n)} \right)^{C-A-B+I} \right\|^{\frac{1}{m+n}}.
\end{aligned} \tag{2.2}$$

If μ, m and n are positive numbers, we can write

$$m = \mu n. \tag{2.3}$$

Using (2.3) in (2.2), we have

$$\begin{aligned}
\frac{1}{R} \leq \lim_{n\to\infty} \sup & \left\| \left(I + \frac{A-I}{l(\mu n,n)} \right)^{l(\mu n,n)I} \left(I + \frac{B-I}{l(\mu n,n)} \right)^{l(\mu n,n)I} \right. \\
& \left. \left(I + \frac{C-I}{l(\mu n,n)} \right)^{-l(\mu n,n)I} \left(\frac{e}{l(\mu n,n)} \right)^{C-A-B+I} \right\|^{\frac{1}{(\mu+1)n}} \leq 1.
\end{aligned}$$

This result is summarized in the following.

Theorem 2.1. *Let A, B and C be matrices in $\mathbb{C}^{N\times N}$ such that $C + l(m,n)I$ is an invertible matrix for all integers $l(m,n) \geq 0$. Let z and w be two complex variables. Then, the radius of convergence of the $l(m,n)$-hypergeometric matrix function is one.*

Here, the more elementary properties of the $l(m,n)$-hypergeometric matrix function are developed and they enjoy a differential operator, which is obtained from Theorem 2.2 and with the help of

Definition (2.1). Also, some matrix recurrence relations for the $l(m,n)$-hypergeometric matrix function are given.

In this connection, the infinitely various recurrence relations are given as follows, directly by increasing or decreasing one in original relation or interesting ones as follows

$$\begin{aligned} {}_2F_1^{l(m,n)}(A+,B;C;z,w) &= \sum_{l(m,n)\geq 0} \frac{(A+I)_{l(m,n)}(B)_{l(m,n)}[(C)_{l(m,n)}]^{-1}}{(l(m,n))!} z^m w^n \\ &= A^{-1} \sum_{l(m,n)\geq 0} (A+l(m,n)I)U_{m,n}(z,w). \end{aligned} \tag{2.4}$$

Similarly, we have

$$\begin{aligned} {}_2F_1^{l(m,n)}(A-) &= (A-I)\sum_{l(m,n)\geq 0} [(A+(l(m,n)-1)I)]^{-1}U_{m,n}(z,w), \\ {}_2F_1^{l(m,n)}(B+) &= B^{-1}\sum_{l(m,n)\geq 0} (B+l(m,n)I)U_{m,n}(z,w), \\ {}_2F_1^{l(m,n)}(B-) &= (B-I)\sum_{l(m,n)\geq 0} [(B+(l(m,n)-1)I)]^{-1}U_{m,n}(z,w), \\ {}_2F_1^{l(m,n)}(C+) &= C\sum_{l(m,n)\geq 0} [(C+l(m,n)I)]^{-1}U_{m,n}(z,w), \\ {}_2F_1^{l(m,n)}(C-) &= (C-I)^{-1}\sum_{l(m,n)\geq 0} (C+(l(m,n)-1)I)U_{m,n}(z,w). \end{aligned} \tag{2.5}$$

For all integers $k \geq 1$, we deduce that

$$\begin{aligned} {}_2F_1^{l(m,n)}(A+kI) &= \prod_{r=1}^{k}(A+(r-1)I)^{-1}\sum_{l(m,n)\geq 0}\prod_{r=1}^{k}(A+(l(m,n)+(r-1))I)U_{m,n}(z,w), \\ {}_2F_1^{l(m,n)}(A-kI) &= \prod_{r=1}^{k}(A-rI)\sum_{l(m,n)\geq 0}\prod_{r=1}^{k}(A+(l(m,n)-r)I)^{-1}U_{m,n}(z,w), \\ {}_2F_1^{l(m,n)}(B+kI) &= \prod_{r=1}^{k}(B+(r-1)I)^{-1}\sum_{l(m,n)\geq 0}\prod_{r=1}^{k}(B+(l(m,n)+(r-1))I)U_{m,n}(z,w), \\ {}_2F_1^{l(m,n)}(B-kI) &= \prod_{r=1}^{k}(B-rI)\sum_{l(m,n)\geq 0}\prod_{r=1}^{k}(B+(l(m,n)-r)I)^{-1}U_{m,n}(z,w), \\ {}_2F_1^{l(m,n)}(C+kI) &= \prod_{r=1}^{k}(C+(r-1)I)\sum_{l(m,n)\geq 0}\prod_{r=1}^{k}(C+(l(m,n)+(r-1))I)^{-1}U_{m,n}(z,w) \end{aligned}$$

and

$${}_2F_1^{l(m,n)}(C-kI) = \prod_{r=1}^{k}(C-rI)^{-1}\sum_{l(m,n)\geq 0}\prod_{r=1}^{k}(C+(l(m,n)-r)I)U_{m,n}(z,w).$$

Here we note that the the differential operator defined as

$$\mathbb{D} = \frac{1}{2}(D)_2 + d_2 = \frac{1}{2}D(D+1) + d_2 = \frac{1}{2}(D^2+D) + d_2, \tag{2.6}$$

where $D = d_1 + d_2$, $d_1 = z\frac{\partial}{\partial z}$ and $d_2 = w\frac{\partial}{\partial w}$.

On using the differential operator (2.6) and applying to the $l(m,n)$-hypergeometric matrix function of (2.1), we get

$$\begin{aligned}\mathbb{D}\ {}_2F_1^{l(m,n)}(A,B;C;z,w) &= \left[\frac{1}{2}(D^2+D)+d_2\right]\ {}_2F_1^{l(m,n)}(A,B;C;z,w)\\ &= \sum_{l(m,n)\geq 0}\frac{\left[\frac{1}{2}((m+n)^2+(m+n))+n\right](A)_{l(m,n)}(B)_{l(m,n)}[(C)_{l(m,n)}]^{-1}}{(l(m,n))!}z^m w^n\\ &= \sum_{l(m,n)\geq 0}\frac{\left[\frac{1}{2}((m+n)(m+n+1))+n\right](A)_{l(m,n)}(B)_{l(m,n)}[(C)_{l(m,n)}]^{-1}}{(l(m,n))!}z^m w^n\\ &= \sum_{l(m,n)\geq 0}\frac{l(m,n)(A)_{l(m,n)}(B)_{l(m,n)}[(C)_{l(m,n)}]^{-1}}{(l(m,n))!}z^m w^n\end{aligned}$$

$$\begin{aligned}&= \sum_{l(m,n)\geq 1}\frac{(A)_{l(m,n)}(B)_{l(m,n)}[(C)_{l(m,n)}]^{-1}}{(l(m,n)-1)!}z^m w^n\\ &= \sum_{l(m,n)\geq 0}\frac{(A)_{l(m,n)+1}(B)_{l(m,n)+1}[(C)_{l(m,n)+1}]^{-1}}{(l(m,n))!}z^{m-1}w^{n+1}\\ &= \sum_{l(m,n)\geq 0}\frac{(A+l(m,n)I)(B+l(m,n)I)[(C+l(m,n)I)]^{-1}(A)_{l(m,n)}(B)_{l(m,n)}[(C)_{l(m,n)}]^{-1}}{(l(m,n))!}z^{m-1}w^{n+1}\\ &= \frac{w}{z}B\ {}_2F_1^{l(m,n)}(B+)+\frac{w}{z}(A-C)B[(C)]^{-1}\ {}_2F_1^{l(m,n)}(C+)+\frac{w}{z}(A-C)\ {}_2F_1-\frac{w}{z}(A-C)\ {}_2F_1^{l(m,n)}(C+)\end{aligned}$$

i.e., the $l(m,n)$-hypergeometric matrix function is a solution to the matrix partial differential equation

$$\left(\mathbb{D}I-\frac{w}{z}(A-C)\right)\ {}_2F_1^{l(m,n)}-\frac{w}{z}B\ {}_2F_1^{l(m,n)}(B+)-\frac{w}{z}(A-C)(B-C)[(C)]^{-1}\ {}_2F_1^{l(m,n)}(C+)=\mathbf{0}. \quad (2.7)$$

On the other hand, the relation (2.7) can be written in the form

$$\mathbb{D}\ {}_2F_1^{l(m,n)}(A,B;C;z,w)=\frac{wAB[(C)]^{-1}}{z}\ {}_2F_1^{l(m,n)}(A+I,B+I;C+I;z,w). \quad (2.8)$$

It is easy to see that the $l(m,n)$-hypergeometric matrix function satisfies matrix partial differential recurrence relation

$$\mathbb{D}(\mathbb{D}-1)\ {}_2F_1^{l(m,n)}(A,B;C;z,w)=\frac{w^2}{z^2}(A)_2(B)_2[(C)_2]^{-1}\ {}_2F_1^{l(m,n)}(A+2I,B+2I;C+2I;z,w). \quad (2.9)$$

From (2.1) and (2.5), we get

$$\begin{aligned}(\mathbb{D}\ I+C-I)\ {}_2F_1^{l(m,n)}(A,B;C;z,w) &= \sum_{l(m,n)\geq 0}\frac{(C+(l(m,n)-1)I)(A)_{l(m,n)}(B)_{l(m,n)}[(C)_{l(m,n)}]^{-1}}{(l(m,n))!}z^m w^n\\ &= \sum_{l(m,n)\geq 0}\frac{(C-I)(A)_{l(m,n)}[(C-I)_{l(m,n)}]^{-1}}{(l(m,n))!}z^m w^n \quad (2.10)\\ &= (C-I)\ {}_2F_1^{l(m,n)}(C-).\end{aligned}$$

From (2.10), we can write

$$\mathbb{D}\ {}_2F_1^{l(m,n)}(A,B;C;z,w)=(C-I)\left[{}_2F_1^{l(m,n)}(A,B;C-I;z,w)-{}_2F_1^{l(m,n)}(A,B;C;z,w)\right] \quad (2.11)$$

and again

$$\mathbb{D}^2 \ {}_2F_1^{l(m,n)}(A,B;C;z,w) = (C-I)(C-2I) \ {}_2F_1^{l(m,n)}(A,B;C-2I;z,w) - \left[(C-I)(C-2I) + (C-I)^2\right] \ {}_2F_1^{l(m,n)}(A,B;C-I;z,w) + (C-I)^2 \ {}_2F_1^{l(m,n)}(A,B;C;z,w). \tag{2.12}$$

Thus, by mathematical induction, we have the following general form :

$$\begin{aligned}
&\Xi(\mathbb{D}) \ {}_2F_1^{l(m,n)}(A,B;C;z,w) = (1+\sum_{k=1}^{N}\mathbb{D}^k) \ {}_2F_1^{l(m,n)}(A,B;C;z,w) \\
&= {}_2F_1^{l(m,n)}(A,B;C;z,w) + \sum_{k=1}^{N}\Big[\prod_{j=1}^{k}(C-jI) \ {}_2F_1^{l(m,n)}(A,B;C-kI;z,w) \\
&-\Big(\prod_{j=1}^{k}(C-jI) + \prod_{j=1}^{k-1}(C-jI)\sum_{k=1}^{k-1}(C-jI)\Big) \ {}_2F_1^{l(m,n)}(A,B;C-(k-1)I;z,w) \\
&+\Big[\prod_{j=1}^{k-1}(C-jI)\sum_{j=1}^{k-1}(C-jI) + \prod_{j=1}^{k-2}(C-jI)\Big(\big(\sum_{j=1}^{k-2}(C-jI)\big)^2 \\
&+\sum_{j=1}^{k-3}(C-jI)(C-(j+1)I) + \sum_{j=1}^{k-4}(C-jI)(C-(j+2)I)...\Big)\Big] \\
&{}_2F_1^{l(m,n)}(A,B;C-(k-2)I;z,w) \\
&-\Big[\prod_{j=1}^{k-1}(C-jI)\sum_{j=1}^{k-1}(C-jI) + \prod_{j=1}^{k-2}(C-jI)\Big(\big(\sum_{j=1}^{k-2}(C-jI)\big)^2 \\
&-\sum_{j=1}^{k-3}(C-jI)(C-(j+1)I) - \sum_{j=1}^{k-4}(C-jI)(C-(j+2)I)\Big) - ...\Big] \\
&+\prod_{j=1}^{k-3}(C-jI)\Big(\big(\sum_{j=1}^{k-3}(C-jI)\big)^3 \\
&-2\sum_{j=1}^{k-4}(C-jI)^2(C-(j+1)I) - 2\sum_{j=1}^{k-4}(C-jI)(C-(j+1)I)^2 \\
&-2\sum_{j=1}^{k-5}(C-jI)^2(C-(j+2)I) - 2\sum_{j=1}^{k-5}(C-jI)(C-(j+1)I)^2 - ...\Big)\Big] \\
&{}_2F_1^{l(m,n)}(A,B;C-(k-3)I;z,w) + ... + (-1)^k(C-I)^k \ {}_2F_1^{l(m,n)}(A,B;C;z,w)\Big],
\end{aligned} \tag{2.13}$$

where N is a finite positive integer. Summarizing, the following result has been established.

Theorem 2.2. *Let A, B and C be commutative matrices in $\mathbb{C}^{N\times N}$ such that $C + l(m,n)I$ is an invertible matrix for all integers $l(m,n) \geq 0$. Then the differential operator (2.13) holds true.*

As a matter of fact, we prove the radius of regularity for the $l(m,n)$-hypergeometric matrix function of two complex variables. We get matrix recurrence relations and study the effect of differential operator $\alpha(\mathbb{D})$ on this function. Note that a more detailed study of all these problems is located in [11] and [12].

3 $p\,l(m,n)$-Hypergeometric Matrix Function of Two Complex Variables

Suppose that p is a positive integer. We define the $p\,l(m,n)$-hypergeometric matrix function ${}^{p}_{2}F_{1}^{l(m,n)}(A,B;C;z,w)$ of two complex variables in the form :

$$\begin{aligned} {}^{p}_{2}F_{1}^{l(m,n)}(A,B;C;z,w) &= \sum_{l(m,n)\geq 0} \frac{(A)_{l(m,n)}(B)_{l(m,n)}[(C)_{l(m,n)}]^{-1}}{(p\,l(m,n))!} z^m w^n \\ = \sum_{l(m,n)\geq 0} W_{m,n}(z,w) z^m w^n, \end{aligned} \tag{3.1}$$

where $W_{m,n} = \frac{(A)_{l(m,n)}(B)_{l(m,n)}[(C)_{l(m,n)}]^{-1}}{(p\,l(m,n))!}$.

Similarly, we give the radius of regularity R for ${}^{p}_{2}F_{1}^{l(m,n)}(A,B;C;z,w)$ and $1 \leq \sigma_{m,n} \leq 2^{\frac{m+n}{2}}$, we have

$$\begin{aligned}
\frac{1}{R} &= \lim_{m+n\to\infty} \sup \left(\frac{\|W_{m,n}\|}{\sigma_{m,n}} \right)^{\frac{1}{m+n}} = \lim_{m+n\to\infty} \sup \left(\frac{\|(A)_{l(m,n)}(B)_{l(m,n)}[(C)_{l(m,n)}]^{-1}\|}{(p\,l(m,n))!\sigma_{m,n}} \right)^{\frac{1}{m+n}} \\
&= \lim_{m+n\to\infty} \sup \left\| \sqrt{2\pi(A+(l(m,n)-1)I)} \left(\frac{(A+(l(m,n)-1)I)}{e} \right)^{(A+(l(m,n)-1)I)} \right\|^{\frac{1}{m+n}} \\
&\quad \left\| \sqrt{2\pi(B+(l(m,n)-1)I)} \left(\frac{(B+(l(m,n)-1)I)}{e} \right)^{(B+(l(m,n)-1)I)} \right\|^{\frac{1}{m+n}} \\
&\quad \left\| \left[\sqrt{2\pi(C+(l(m,n)-1)I)} \left(\frac{(C+(l(m,n)-1)I)}{e} \right)^{(C+(l(m,n)-1)I)} \right]^{-1} \right\|^{\frac{1}{m+n}} \\
&\quad \left\| \left[\sqrt{2p\pi l(m,n)} \left(\frac{p\,l(m,n)}{e} \right)^{p\,l(m,n)} \right]^{-1} \frac{1}{\sigma_{m,n}} \right\|^{\frac{1}{m+n}} \\
&\leq \lim_{m+n\to\infty} \sup \left\| \left(\frac{(A+(l(m,n)-1)I)}{e} \right)^{A-I} \left(\frac{(A+(l(m,n)-1)I)}{e} \right)^{l(m,n)I} \right\|^{\frac{1}{m+n}} \\
&\quad \left\| \left(\frac{(B+(l(m,n)-1)I)}{e} \right)^{B-I} \left(\frac{(B+(l(m,n)-1)I)}{e} \right)^{l(m,n)I} \right\|^{\frac{1}{m+n}} \\
&\quad \left\| \left(\frac{(C+(l(m,n)-1)I)}{e} \right)^{-C+I} \left(\frac{(C+(l(m,n)-1)I)}{e} \right)^{-l(m,n)I} \left(\frac{p\,l(m,n)}{e} \right)^{-p\,l(m,n)} \right\|^{\frac{1}{m+n}} \\
&\leq \lim_{m+n\to\infty} \sup \left\| \left(\frac{(A+(l(m,n)-1)I)}{l(m,n)} \right)^{l(m,n)I} \left(\frac{(B+(l(m,n)-1)I)}{l(m,n)} \right)^{l(m,n)I} \right\|^{\frac{1}{m+n}} \\
&\quad \left\| \left(\frac{(C+(l(m,n)-1)I)}{l(m,n)} \right)^{-l(m,n)I} \left(\frac{e}{p\,l(m,n)} \right)^{(p-1)\,l(m,n)} \right\|^{\frac{1}{m+n}} \\
&\leq \lim_{m+n\to\infty} \sup \left\| \left(I + \frac{A-I}{l(m,n)} \right)^{l(m,n)I} \left(I + \frac{B-I}{l(m,n)} \right)^{l(m,n)I} \right\|^{\frac{1}{m+n}} \\
&\quad \left\| \left(I + \frac{C-I}{l(m,n)} \right)^{-l(m,n)I} \left(\frac{e}{p\,l(m,n)} \right)^{(p-1)\,l(m,n)} \right\|^{\frac{1}{m+n}} = 0,\ p > 1.
\end{aligned}$$

This result is summarized below.

Theorem 3.1. *Let A, B and C be matrices in $\mathbb{C}^{N\times N}$ such that the condition $C + l(m,n)I$ is an invertible matrix for every integer $l(m,n) \geq 0$. Then, the $p\,l(m,n)$-hypergeometric matrix function is an entire function if $p > 1$.*

Putting $p = 1$ in (3.1), we have

$$\frac{1}{R} = \lim_{m+n\to\infty} \sup \left(\frac{\|(A)_{l(m,n)}(B)_{l(m,n)}[(C)_{l(m,n)}]^{-1}\|}{(l(m,n))!\sigma_{m,n}} \right)^{\frac{1}{m+n}} \leq 1$$

i.e., the radius of convergence of the $l(m,n)$-hypergeometric matrix function is one.

Similarly, for the $p\ l(m,n)$-hypergeometric matrix function is a solution of the matrix partial differential equation

$$\mathbb{D}\ {}_2^pF_1^{l(m,n)}(A,B;C;z,w) - \frac{1}{p}\left(\frac{w}{z}\right)^{\frac{1}{p}}(A)(B)[(C)]^{-1}\ {}_2^pF_1^{l(m,n)}(A+\frac{1}{p}I, B+\frac{1}{p}I; C+\frac{1}{p}I; z,w) = \mathbf{0}. \quad (3.2)$$

In the same way, we apply by using the differential operator on the power series ${}_2^pF_1^{l(m,n)}(A,B;C;z,w)$, then we have

$$\begin{aligned}
&\left[\mathbb{D}(\mathbb{D}-\frac{1}{p})(\mathbb{D}-\frac{2}{p})...(\mathbb{D}-\frac{p-1}{p})(\mathbb{D}\ I + C - I)\right]\ {}_2^pF_1^{l(m,n)}(A,B;C;z,w)\\
&= \sum_{l(m,n)\geq 0} \frac{l(m,n)[l(m,n)-\frac{1}{p}][l(m,n)-\frac{2}{p}]...[l(m,n)-\frac{p-1}{p}](C+(l(m,n)-1)I)(A)_{l(m,n)}[(C)_{l(m,n)}]^{-1}}{(p\,l(m,n))!} z^m w^n\\
&= \frac{1}{p^p}\sum_{l(m,n)\geq 0} \frac{(C+(l(m,n)-1)I)(A)_{l(m,n)}(B)_{l(m,n)}[(C)_{l(m,n)}]^{-1}}{(p\,l(m,n)-p)!} z^m w^n\\
&= \frac{1}{p^p}\sum_{l(m,n)\geq 0} \frac{(C+l(m,n)I)(A)_{l(m,n)+1}(B)_{l(m,n)+1}[(C)_{l(m,n)+1}]^{-1}}{(p\,l(m,n))!} z^{m-1} w^{n+1}\\
&= \frac{w}{z}\frac{1}{p^p}\sum_{l(m,n)\geq 0} \frac{(C+l(m,n)I)(A)_{l(m,n)+1}(B)_{l(m,n)+1}[(C)_{l(m,n)+1}]^{-1}}{(p\,l(m,n))!} z^m w^n\\
&= \frac{w}{z}\frac{1}{p^p}\sum_{l(m,n)\geq 0} \frac{[A+l(m,n)I][B+l(m,n)I](A)_{l(m,n)}(B)_{l(m,n)}[(C)_{l(m,n)}]^{-1}}{(p\,l(m,n))!} z^m w^n\\
&= \frac{w}{z}\frac{1}{p^p}\left[AB + \mathbb{D}(A+B+I) + \mathbb{D}(\mathbb{D}-1)I\right]\ {}_2^pF_1^{l(m,n)}(A,B;C;z,w).
\end{aligned}$$

Therefore, the following result has been established.

Theorem 3.2. *Let A, B and C be matrices in $\mathbb{C}^{N\times N}$ such that the condition $C + l(m,n)I$ is an invertible matrix for every integer $l(m,n) \geq 0$. Let z and w be two complex variables. Then the $p\,l(m,n)$-hypergeometric matrix function is a solution of this matrix partial differential equation*

$$\begin{aligned}
&\left[\mathbb{D}(\mathbb{D}-\frac{1}{p})(\mathbb{D}-\frac{2}{p})...(\mathbb{D}-\frac{p-1}{p})(D\ I + C - I)\right.\\
&\left.-\frac{w}{z}\frac{1}{p^p}\left(AB + \mathbb{D}(A+B+I) + \mathbb{D}(\mathbb{D}-1)I\right)\right]\ {}_2^pF_1^{l(m,n)}(A,B;C;z,w) = \mathbf{0}.
\end{aligned} \quad (3.3)$$

As a matter of fact, we define $p\ l(m,n)$ hypergeometric matrix function and get the radius of regularity and matrix partial differential equation. We cannot detail them here. Most these results are located in [11] and [12].

Conclusion

In conclusion, we define of the $l(m,n)$-hypergeometric and $p\ l(m,n)$-hypergeometric matrix functions of two complex variables, more results could be obtained the radius of regularity, matrix recurrence relations, matrix partial differential equation and the differential operator $\alpha(\mathbb{D})$, but the details are omitted for reasons of brevity and clearly some directions to develop more researches and studies in that area. Note that a more detailed study are located in [11] and [12]. The results of this paper are original, variant, significant progress and so it is interesting and capable to develop its study in the future. Further results and applications will be discussed in a forthcoming paper.

In a forthcoming work, we will consider the problems of a unified approach to the theory of new special matrix functions by the technique discussed in this paper.

Definition 3.1. Let A, B and C be commutative matrices in $C^{N\times N}$ such that $C + l_3(m,n)I$ is an invertible matrix for all integers $l_3(m,n) \geq 0$. We will define the generalized $l(m,n)$-hypergeometric matrix function of two complex variables in the form

$$F^{l_1(m,n),l_2(m,n)}_{l_3(m,n),l(m,n)}\left(A, B; C; z, w\right) = \sum_{l(m,n)\geq 0} \frac{(A)_{l_1(m,n)}(B)_{l_2(m,n)}[(C)_{l_3(m,n)}]^{-1}}{(l(m,n))!} z^m w^n.$$

Open problem

One can use the same class of new integral representation and operational methods for the generalized hypergeometric matrix functions $F^{l_1(m,n),l_2(m,n)}_{l_3(m,n),l(m,n)}\left(A, B; C; z, w\right)$, from which a variety of interesting results follow as special cases and clearly some directions to develop more researches and studies in that area. Also discuss its many particular cases; e. g. Kampé de Fériet matrix functions, Appells matrix functions etc. Hence, new results and further applications can be obtained. Further results and applications will be discussed in a forthcoming work.

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Competing Interests

The author declares that no competing interests exist.

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