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On Stochastic Volatility in the Valuation of European Options

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Abstract

This paper presents stochastic volatility in the valuation of European options. Stochastic volatility models treat the volatility of the underlying asset as a random process rather than the constant volatility assumption of the Black-Scholes model. By changing the model parameters, almost all kinds of asset distributions can be generated by a negative correlation between the stock price process and the volatility process. It is observed that an asset's log-return distribution is non-Gaussian which is characterized by heavy tails and high peaks. Heston model presents a new approach for a closed form valuation of options specifying the dynamics of the squared volatility as a square-root process and applying Fourier inversion techniques for the pricing procedure. Determination of the market growth rate of the stock share was considered. We also considered the effect of volatility and correlation parameter on the kurtosis and skewness of the density function.

Keywords: European option, Heston model, Nigerian stock exchange, stochastic volatility model.

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1 Introduction

Derivative contracts are securities whose values are contingent on some other financial instruments or variable, called the underlier [1]. Volatility derivatives are products with volatility as the main underlying notion. These products are particularly important for market investors, as they are used to gain insight into the level of volatility in order to manage the market volatility risk [2].

The pricing methodologies proposed by [3,4] are the most significant and influential development in option pricing theory. However, the assumptions underlying the original works were questioned and became the subject of a wide theoretical and empirical study. Soon it became clear that extensions are necessary in order to fit the empirical data. Black and Scholes demonstrate how to price options under this assumption. Today this model is known as the Black-Scholes model and remains one of the most successful and widely used derivatives pricing models available. The main drawback of the Black-Scholes model is the rather strong assumption that the volatility of stock returns is constant. Under the assumption, when the implied volatility calculated from the empirical option data is plotted against option's strike price and time to maturity, the resulting graph should be a flat surface. However, in practice, the implied volatility surface is not flat and the implied volatility tends to vary with the strike price and time to maturity. This disparity is known as the volatility skew [5]. This consequently leads to development of dynamic volatility modeling. A natural extension is so-called stochastic volatility model in which the volatility is a function of some stochastic variables [5]. The story of modeling financial markets with stochastic processes dates back to 1900 with research of Bachelier. He modeled the stock prices as a Brownian motion with drift. A more appropriate model is based on geometric Brownian motion [5].

There has been vast work on option pricing since the appearance of the celebrated Black and Scholes formula. The foundation of all these recent techniques had been laid long time before by Charles Castelli, who in 1877 talked about the different purposes of options in his book titled "The Theory of Options in Stocks and Shares". The first known analytical valuation for options was presented in 1897 by Louis Bachelier in his mathematics dissertation "Theorie de la Speculation". The pitfalls in his work were that the process he chose generated negative security prices and the option prices were in some cases greater than the prices of the underlying assets.

Since the Black-Scholes formula was derived, a number of empirical studies have concluded that the assumption of constant volatility is inadequate to describe the stock returns. The volatility has been observed to exhibit consistently some empirical characteristics [5]:

- Volatility tends to revert around some long term value;
- Volatility clusters with time: large (small) volatility tend to follow large (small) price changes;
- Volatility is correlated with stock returns.

The stochastic volatility models have been put forward to model the variability of volatility and to capture the volatility skew [5].

One option valuation problem that has hitherto remained unsolved is the pricing of European call on a stock that has a stochastic volatility [6]. From the work of [7], the differential equation that the option must satisfy is known. The solution of this differential equation is independent of risk preferences if the volatility is a traded asset or the volatility is uncorrelated with aggregate consumption. If either of these conditions holds, the risk neutral valuation arguments of [8] can be used in a straightforward way.

[9] considered the pricing of options on asset with stochastic volatilities; in their paper they produced a solution series form for the situation in which the stock price is instantaneously uncorrelated with the volatility. They did not assume that the volatility is a traded asset and a constant correlation between the instantaneous rate of change of the volatility and the rate of change of aggregate consumption can be accommodated. They also found that the Black-Scholes price frequently overprices options and that the degree of overpricing increases with time to maturity. The option price is lower than the Black-Scholes (B-S) price when the option is close to being at the money and higher when it is deep in or out of the money. The exercise price for which overpricing by Black-Scholes takes place is within about ten percent of the security price. This is the range of the exercise prices over which most option trading takes place. This effect is exaggerated as the time to maturity increases. The longer the time to maturity, the lower the implied volatility.

The most popular stochastic volatility model was introduced by Heston. In his influential paper he presents a new approach for a closed form valuation of options specifying the dynamics of the squared volatility (variance) as a square-root process and applying Fourier inversion techniques for the pricing procedure. The characteristic function approach of this model turned out to be a very powerful tool.

According to [10], in a stochastic-volatility model, volatility is driven by a random source that is different from the random source that drives the asset return process, even when the two random sources may be correlated. In contrast to a deterministic-volatility model in which the investor incurs only the risk from a randomly evolving asset price, in a stochastic-volatility environment, an investor in the options market bears the additional risk of a randomly evolving volatility. In a deterministic volatility model, an investor can hedge the risk from the asset price by trading an option and a risk-free asset based on a risk exposure computed using an option pricing formula**.** They also gave an overview of different specifications of asset price volatility that are widely used in option pricing models.

Stochastic volatility models assume that volatility itself is a random process and fluctuates over time. Stochastic volatility models were studied by [11,12,13]. Other models for the volatility dynamics were proposed by [14,15,16,17,18,19,20] just to mention a few. In all these models the stochastic process governing the asset price dynamics is driven by a subordinated stochastic volatility process that may or may not be independent [21].

In this paper, we shall investigate Heston model and its accuracy for the valuation of options under stochastic volatility. The paper is structured as follows; Section 2 provides the theoretical background on which the models rely on and the derivation of Heston model. Numerical calculations are made in Section 3, where the results of the tests are presented as well as comments on the results. Section 4 concludes the paper.

2 Stochastic Volatility

Volatility refers to the amount of uncertainty or risk about the size of changes in security's value. This is also the key to understanding why option prices fluctuate and act the way they do. In fact, volatility is the most important concept in the valuation of options and it is denoted by σ .

A model of volatility is needed for managing portfolios containing options (including derivatives and other securities containing options) for which market quotes are not readily available and that consequently must be marked to model rather than marked to market. Accurate assessments of volatility are also key inputs into the construction of hedges, which limit risk exposures, for such portfolios. Because of the central role that volatility plays in derivative valuation and hedging, a substantial literature is devoted to the specification of volatility and its measurement. Modeling volatility is challenging because volatility in financial and commodity markets appears to be highly unpredictable. There has been a proliferation of volatility specifications since the original, simple constant-volatility assumption of the famous option pricing model developed by Fischer Black and Myron S. Scholes.

Volatility is one of the factors affecting option pricing. The greater the expected movement in the price of the underlying assets due to high volatility, the greater the probability that the option can be exercised for a profit and hence the more valuable is the option [22].

A higher volatility means that a security's value can potentially be spread out over a larger range of values. This means that the price of security can change dramatically over a short time period in either direction.

A lower volatility means that a security's value does not fluctuate dramatically but changes in value at a steady pace over a period of time. For options, volatility is ‘good' because the greater the volatility of the underlying, the greater the value of the option while for some financial derivatives, volatility is `bad'. This is due to the fact that the purchases of options enjoy only the upside potential not downside risk. Other financial derivatives have both risks.

There are two types of volatility namely implied and historical volatilities [22]. Implied volatility measures the price movement of the option itself. Historical volatility is also called statistical volatility that measures the rate of movement in the price of the underlying asset.

2.1 Stochastic Volatility Model

The popularity of stochastic volatility in option pricing grew out of the fact that distributions of the asset returns exhibit fatter tails than those of the normal distribution [23,24]. Thus, the observed frequency of extreme asset returns in fat tail distribution is much higher than would occur if returns were described by a normal distribution. Stochastic volatility models can be consistent with fat tails of the return distribution. The occurrence of fat tails would imply, for example, that out-of-the-money options would be underpriced by the Black-Scholes model, which assumes that returns are normally distributed.

One problem arising with the assumptions of a model such as the Black-Scholes is volatility smile (or volatility skew in some markets). If we consider options on an underlying equity with different strike prices, then the volatilities implied by their market prices should be the same. They measure the risk for the same underlying asset. Similar patterns are found by altering time-length to maturity when the market prices are used to find the implied volatilities. These patterns are very difficult to explain in a Black-Scholes world. Both constant- and stochastic volatility models assume the stock price follows a stochastic process. A general representation of the continuous-type stochastic volatility model is given by the stochastic differential equation:

$$\left.\begin{aligned} dS_t &= \mu S_t dt + \sigma_t S dW_t^1 \\ \sigma_t &= f(Y_t) \end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} dY_t &= a(t, Y_t)dt + b(t, Y_t)dW_t^2 \\ dW_t^1 dW_t^2 &= \rho dt \end{aligned}\right\} \tag{2}$$

Here S_t is called the stock price at time t, the variable μ is the drift or expected rate of return and σ_t is the volatility of the stock price. W_t^1 and W_t^2 are two correlated standard geometric Brownian motion which can be defined as the process for the price of the underlying asset. By dynamics of historical prices, we assume that the volatility follows a stochastic process.

The constant parameter ρ is the correlation coefficient between these two Brownian motions dW_t^1 and dW_t^2 in (1) and (2) respectively. We can also express W_t^2 as

$$W_t^2 = \rho W_t^1 + \sqrt{1-\rho^2}\,W_t \tag{3}$$

where W_t is a standard Brownian motion independent of W_t^1. There are some economic arguments for a negative correlation between stock price and volatility shocks.

<u>2.1.1 The valuation of options in the stochastic volatility model</u>

To price options in the stochastic volatility model, we shall apply no-arbitrage arguments method. The riskless portfolio is constructed as in the Black-Scholes model but the construction method is different. In the stochastic volatility option valuation model, there is only one traded risky asset S but two random sources W_t^1 and W_t^2. So the market is incomplete. Since no-arbitrage arguments are enough to give the option price, we need additional assumptions. In the following derivation, equilibrium arguments are also employed. The market is complete when we have two traded assets, the underlying asset S and a benchmark option G. Then all other options can be replicated by these two traded assets. A riskless portfolio θ consists of an option F which we want to price, $-\Delta_1$ shares of the underlying asset S and $-\Delta_2$ shares of the benchmark option G. Then we have that

$$\theta = F - \Delta_1 S - \Delta_2 G \tag{4}$$

The portfolio is self-financing, so that

$$d\theta = dF - \Delta_1 dS - \Delta_2 dG \tag{5}$$

F and G are functions of variables $t, S = S_t$ and $Y = Y_t$. By applying the two-dimensional It$\widehat{o}$'s formula, we have for dF and dG respectively as follows

$$dF = \left[\frac{\partial F}{\partial t} + \frac{1}{2} f(Y)^2 S^2 \frac{\partial^2 F}{\partial S^2} + \frac{1}{2} b^2 \frac{\partial^2 F}{\partial Y^2} + \rho f(Y) Sb \frac{\partial^2 F}{\partial S \partial Y}\right] dt + \frac{\partial F}{\partial Y} dY + \frac{\partial F}{\partial S} dS \tag{6}$$

$$dG = \left[\frac{\partial G}{\partial t} + \frac{1}{2} f(Y)^2 S^2 \frac{\partial^2 G}{\partial S^2} + \frac{1}{2} b^2 \frac{\partial^2 G}{\partial Y^2} + \rho f(Y) Sb \frac{\partial^2 G}{\partial S \partial Y}\right] dt + \frac{\partial G}{\partial Y} dY + \frac{\partial G}{\partial S} dS \tag{7}$$

Substituting (6) and (7) into (5), we have

$$\left.\begin{aligned} d\theta = &\left[\frac{\partial F}{\partial t} + \frac{1}{2} f(Y)^2 S^2 \frac{\partial^2 F}{\partial S^2} + \frac{1}{2} b^2 \frac{\partial^2 F}{\partial Y^2} + \rho f(Y) Sb \frac{\partial^2 F}{\partial S \partial Y}\right] dt + \frac{\partial F}{\partial Y} dY + \frac{\partial F}{\partial S} dS - \Delta_1 dS \\ -\Delta_2 &\left\{\left[\frac{\partial G}{\partial t} + \frac{1}{2} f(Y)^2 S^2 \frac{\partial^2 G}{\partial S^2} + \frac{1}{2} b^2 \frac{\partial^2 G}{\partial Y^2} + \rho f(Y) Sb \frac{\partial^2 G}{\partial S \partial Y}\right] dt + \frac{\partial G}{\partial Y} dY + \frac{\partial G}{\partial S} dS\right\} \end{aligned}\right\} \tag{8}$$

Therefore, (8) can be written as

$$\left.\begin{aligned} d\theta = &\left[\frac{\partial F}{\partial t} + \frac{1}{2} f(Y)^2 S^2 \frac{\partial^2 F}{\partial S^2} + \frac{1}{2} b^2 \frac{\partial^2 F}{\partial Y^2} + \rho f(Y) Sb \frac{\partial^2 F}{\partial S \partial Y}\right] dt + \\ -\Delta_2 &\left\{\begin{aligned} &\left[\frac{\partial G}{\partial t} + \frac{1}{2} f(Y)^2 S^2 \frac{\partial^2 G}{\partial S^2} + \frac{1}{2} b^2 \frac{\partial^2 G}{\partial Y^2} + \rho f(Y) Sb \frac{\partial^2 G}{\partial S \partial Y}\right] dt + \\ &\left[\frac{\partial F}{\partial S} - \Delta_2 \frac{\partial G}{\partial S} dS - \Delta_1\right] dS + \left[\frac{\partial F}{\partial Y} - \Delta_2 \frac{\partial G}{\partial Y}\right] dY \end{aligned}\right\} \end{aligned}\right\} \tag{9}$$

To make the portfolio riskless, we choose

$$\left.\begin{aligned} &\frac{\partial F}{\partial S} - \Delta_2 \frac{\partial G}{\partial S} - \Delta_1 = 0 \\ &\frac{\partial F}{\partial Y} - \Delta_2 \frac{\partial G}{\partial Y} = 0 \end{aligned}\right\} \tag{10}$$

To eliminate dS and dY terms, solving (10) gives

$$\left.\begin{aligned}\Delta_1 &= \frac{\partial F}{\partial S} - \left(\frac{\partial F}{\partial Y}\Big/\frac{\partial G}{\partial Y}\right)\frac{\partial G}{\partial S} \\ \Delta_2 &= \left(\frac{\partial F}{\partial Y}\Big/\frac{\partial G}{\partial Y}\right)\end{aligned}\right\} \tag{11}$$

The portfolio is riskless if we rebalance it according to (11). On the other hand, the riskless portfolio must earn a risk-free rate; otherwise there would be an arbitrage opportunity. So

$$\left.\begin{aligned}d\theta &= \left[\frac{\partial F}{\partial t} + \frac{1}{2}f(Y)^2S^2\frac{\partial^2 F}{\partial S^2} + \frac{1}{2}b^2\frac{\partial^2 F}{\partial Y^2} + \rho f(Y)Sb\frac{\partial^2 F}{\partial S\partial Y}\right]dt + \\ -\Delta_2 &\left\{\left[\frac{\partial G}{\partial t} + \frac{1}{2}f(Y)^2S^2\frac{\partial^2 G}{\partial S^2} + \frac{1}{2}b^2\frac{\partial^2 G}{\partial Y^2} + \rho f(Y)Sb\frac{\partial^2 G}{\partial S\partial Y}\right]dt\right\}\end{aligned}\right\} \tag{12}$$

Substituting (11) into (12), we have

$$\left.\begin{aligned}d\theta &= \left[\frac{\partial F}{\partial t} + \frac{1}{2}f(Y)^2S^2\frac{\partial^2 F}{\partial S^2} + \frac{1}{2}b^2\frac{\partial^2 F}{\partial Y^2} + \rho f(Y)Sb\frac{\partial^2 F}{\partial S\partial Y}\right]dt + \\ -&\left(\frac{\partial F}{\partial Y}\Big/\frac{\partial G}{\partial Y}\right)\left\{\left[\frac{\partial G}{\partial t} + \frac{1}{2}f(Y)^2S^2\frac{\partial^2 G}{\partial S^2} + \frac{1}{2}b^2\frac{\partial^2 G}{\partial Y^2} + \rho f(Y)Sb\frac{\partial^2 G}{\partial S\partial Y}\right]dt\right\}\end{aligned}\right\} \tag{13}$$

Since the portfolio is riskless then it must earn a return similar to other short term riskless securities such as bank account. Therefore, we have

$$d\theta = r\theta dt \tag{14}$$

Substituting (4) and (13) into (14), yields

$$\left.\begin{aligned}&\left[\frac{\partial F}{\partial t} + \frac{1}{2}f(Y)^2S^2\frac{\partial^2 F}{\partial S^2} + \frac{1}{2}b^2\frac{\partial^2 F}{\partial Y^2} + \rho f(Y)Sb\frac{\partial^2 F}{\partial S\partial Y} - rF + rS\frac{\partial F}{\partial S}\right]\Big/\frac{\partial F}{\partial Y} = \\ &\left[\frac{\partial G}{\partial t} + \frac{1}{2}f(Y)^2S^2\frac{\partial^2 G}{\partial S^2} + \frac{1}{2}b^2\frac{\partial^2 G}{\partial Y^2} + \rho f(Y)Sb\frac{\partial^2 G}{\partial S\partial Y} - rG + rS\frac{\partial G}{\partial S}\right]\Big/\frac{\partial G}{\partial S}\end{aligned}\right\} \tag{15}$$

Notice that the left hand side is a function of F only and right hand side is a function of G only. Let the function $L(t, S, Y)$ which is independent of any particular option be denoted by:

$$L(t,S,Y)=a(t,Y)-b(t,Y)\Omega(t,S,Y) \tag{16}$$

where $a(t,Y)$ is called the drift term of the driving process Y and $\Omega(t,S,Y)$ is called the market volatility risk. $\Omega(t,S,Y)$ cannot be determined by the arbitrage theory alone but rather it is completely determined by the benchmark option G. So the market determines the price of volatility risk.Now setting the left hand side of (15) to $-L(t,S,Y)$, then the equation for the option F is given by

$$\frac{\partial F}{\partial t}+\frac{1}{2}f(Y)^2S^2\frac{\partial^2 F}{\partial S^2}+\frac{1}{2}b^2\frac{\partial^2 F}{\partial Y^2}+\rho f(Y)Sb\frac{\partial^2 F}{\partial S\partial Y}-rF+rS\frac{\partial F}{\partial S}=-L(t,S,Y)\frac{\partial F}{\partial Y} \tag{17}$$

Substituting (16) in (17), we have the partial differential equation for the option of the form

$$\frac{\partial F}{\partial t}+\frac{1}{2}f(Y)^2S^2\frac{\partial^2 F}{\partial S^2}+\frac{1}{2}b^2\frac{\partial^2 F}{\partial Y^2}+\rho f(Y)Sb\frac{\partial^2 F}{\partial S\partial Y}-rF+rS\frac{\partial F}{\partial S}+(a-b\Omega)\frac{\partial F}{\partial Y}=0 \tag{18}$$

Given the terminal condition for F, (18) is solvable under some driving processes.

Remarks: Risk-neutral valuation method can also be applied to the stochastic volatility model above, but the problem here is how to construct equivalent martingale measures and what the asset price process and the volatility process will be under these measures. To change measure from the objective measure P to an equivalent martingale measure Q, we need to use the Gisarnov Theorem.

2.2 Heston's Stochastic Volatility Model

Many traders have used stochastic volatility model for the valuation of options. Heston model is the most popular one among several existed stochastic volatility model. This model assumes that v_t follows a Cox-Ingersoll-Ross (CIR) process of the form

$$\left.\begin{aligned} dv_t &= \kappa(\pi-v_t)dt+\sigma\sqrt{v_t}dW_t \\ f(v_t) &= \sqrt{v_t} \end{aligned}\right\} \tag{19}$$

We rewrite the model in the following way as follows:

$$dS_t=\mu S_t dt+\sqrt{v_t}S_t dW_t^* \tag{20}$$

$$dv_t=\kappa(\pi-v_t)dt+\sigma\sqrt{v_t}dW_t^{**} \tag{21}$$

$$dW_t^* dW_t^{**}=\rho dt \tag{22}$$

Where W_t^* and W_t^{**} are two standard Brownian motions with correlation $\rho \in [-1,1]$, π the long-run variance, κ the rate of mean reversion and σ the volatility. The drift term of the specified process in (21) is an affine function of the state variable itself. The affinity makes the model easier to solve. Here we assumed no dividend paying stock and the interest rate is constant. By the same argument above, the value of any option $U(S,\nu,t)$ must satisfy the equation,

$$\left.\begin{aligned} &\frac{\partial U}{\partial t}+\frac{1}{2}\nu S^2\frac{\partial^2 F}{\partial S^2}+\frac{1}{2}\nu\sigma^2\frac{\partial^2 U}{\partial v^2}+\rho\nu S\sigma\frac{\partial^2 F}{\partial S\partial v}-rU+rS\frac{\partial U}{\partial S}+\\ &\left[\kappa(\pi-\nu)-\Omega(S,\nu,t)\right]\sigma\sqrt{\nu}\frac{\partial U}{\partial \nu}=0,\ \forall\ 0\le t\le T, S>0, v>0 \end{aligned}\right\} \tag{23}$$

$\Omega(S,\nu,t)$ is called the market price of volatility risk. Heston model chooses the market price of volatility risk to be proportional to the volatility, i.e. $\Omega(S,\nu,t)=\kappa\sqrt{\nu}$ or $\Omega(S,\nu,t)\sigma\sqrt{\nu}=\kappa\sigma\nu$. Let $\lambda=\kappa\sigma$, so the coefficient of $\frac{\partial U}{\partial \nu}$ in (23) becomes $\left[\kappa(\pi-\nu)-\lambda\nu\right]$. This choice of market price of volatility risk gives us analytical advantages. It helps the model to have a closed form solution.

2.2.1 A closed form solution

Heston modelis used to solvethis equation for a European option by using a technique based on characteristic functions. This model can also be used to solve some other important models, so we give a brief introduction here. The solution of the equation has the form which is similar to the Black-Scholes formula

$$U(S,\nu,t)=SP_1-Ke^{-r(T-t)}P_2 \tag{24}$$

Suppose

$$z=\ln S \tag{25}$$

and substitute the proposed solution (24) into (23), then we have partial differential equation which P_1 and P_2 should satisfy;

$$\frac{1}{2}\frac{\partial^2 P_j}{\partial z^2}+\rho\sigma\nu\frac{\partial^2 P_j}{\partial z\partial \nu}+\frac{\sigma^2\nu}{2}\frac{\partial^2 P_j}{\partial \nu^2}+(r+u_j\nu)\frac{\partial P_j}{\partial z}+(a-b_j\nu)\frac{\partial P_j}{\partial \nu}+\frac{\partial P_j}{\partial t}=0 \tag{26}$$

For $j=1,2$, where

$$\left.\begin{aligned} &u_1=\frac{1}{2}, u_2=-\frac{1}{2}, a=\kappa\pi,\\ &b_1=\kappa-\rho\sigma+\lambda, b_2=\kappa+\lambda \end{aligned}\right\} \tag{27}$$

The partial differential equation must be solved subject to the terminal condition

$$P_j(z,\nu,T,\ln K)=1_{\{z\geq \ln K\}} \quad (28)$$

They can be interpreted as "risk-neutralized probabilities". The corresponding characteristic functions $f_1(z,\nu,t;\phi)$ and $f_2(z,\nu,t;\phi)$ for risk-neutralized probabilities P_1 and P_2 respectively will satisfy similar partial differential equation (26) with the terminal condition

$$f_j(z,\nu,t;\phi)=e^{i\phi z} \quad (29)$$

We guess a functional form due to linearity of the coefficients

$$f_j(z,\nu,t;\phi)=e^{C_j(T-t;\phi)+D_j(T-t;\phi)+i\phi z} \quad (30)$$

Substituting this functional into (26), we have two ordinary differential equations

$$\frac{1}{2}\phi^2+\rho\sigma\phi_i C_j+\frac{1}{2}\sigma^2 C_j^2+u_j\phi_j-b_jC_j-\frac{\partial C_j}{\partial t}=0 \quad (31)$$

$$r\phi_i+aD_j+\frac{\partial D_j}{\partial t}=0 \quad (32)$$

subject to the terminal conditions given by $C(0;\phi)=0, D(0;\phi)=0$. The solutions to these two ordinary differential equations are

$$C_j(t;\phi)=r\phi_i(T-t)+\frac{a}{\sigma^2}\left\{(b_j-\rho\sigma\phi_i+d_j)(T-t)+2\ln\left[\frac{1-g_je^{d_j(T-t)}}{1-g_j}\right]\right\} \quad (33)$$

$$D_j(t;\phi)=\frac{b_j-\rho\sigma\phi_i+d_j}{\sigma^2}\left[\frac{1-e^{d_j(T-t)}}{1-g_je^{d_j(T-t)}}\right] \quad (34)$$

and

$$\left.\begin{aligned} g_j&=\frac{b_j-\rho\sigma\phi_j+d_j}{b_j-\rho\sigma\phi_j-d_j}\\ d_j&=\sqrt{(\rho\sigma\phi_i-b_j)^2-\sigma^2(2u_j\phi_i-\phi^2)}\end{aligned}\right\} \quad (35)$$

Then the risk-neutral probabilities P_1 and P_2 can be recovered by inverting the corresponding characteristic functions

$$P_j(z,\nu,t;\ln K)=\frac{1}{2}+\frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left[\frac{e^{-i\phi\ln K}f_j(z,\nu,t;\phi)}{i\phi}\right]d\phi \tag{36}$$

The only part that poses a slight problem is the limits of the integral in (36). This integral cannot be evaluated exactly, but can be approximated with reasonable accuracy by using some numerical integration techniques such as Gauss Lagendre or Gauss Lobatto integration.

Hence the price of European call option is given by (24), (30) and (36).

Remarks:

- The price of European put options can be obtained using the put-call parity.
- The essence of the characteristic function methodology is using Fourier series.
- Fast Fourier transform algorithm is introduced by [25]
- Heston model incorporates the stochastic interest rates into the model and apply the stochastic volatility model to bond currency options.

In the sequel we shall give the expressions of the following Greeks namely delta, gamma and Vega respectively as follows

Differentiating (23) with respect to *S* we have:

$$\Delta=\frac{\partial U(S,\nu,t)}{\partial S}=P_1 \tag{37}$$

By differentiating (37) with respect to *S*yields

$$\Gamma=\frac{\partial^2 U(S,\nu,t)}{\partial S^2}=\frac{\partial\Delta}{\partial S}=\frac{\partial\Delta}{\partial z}\frac{\partial z}{\partial S}=\frac{1}{S}\frac{\partial P_1}{\partial z} \tag{38}$$

$$\frac{\partial P_1}{\partial z}=\frac{1}{\pi}\int_0^{\infty}\mathrm{Re}\left[e^{-i\phi\ln K}f_j(z,\nu,t;\phi)\right]d\phi \tag{39}$$

For convenience, in Heston model we define Vega as the first derivative of the option price in(23)with respect to the spot variance

$$\frac{\partial U(S,\nu,t)}{\partial \nu}=S\frac{\partial P_1}{\partial \nu}-Ke^{-r(T-t)}\frac{\partial P_2}{\partial \nu} \tag{40}$$

Where

$$\frac{\partial P_j}{\partial \nu} = \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{D_j(T-t;\phi)e^{-i\phi\ln K} f_j(z,\nu,t;\phi)}{i\phi}\right]d\phi,\ j=1,2 \tag{41}$$

The above formula is quite ‘explicit’ and easy to code in MATLAB.

<u>2.2.2 Parameter estimation</u>

The parameter to be estimated is μ which was defined earlier as the drift parameter in (1). The log of likelihood is given by

$$\frac{\mu}{\sigma}\int_0^T dS_i - \frac{\mu^2}{2\sigma}\int_0^T S_i d_i = \frac{\mu}{\sigma}\left(\left(S(T)-S(0)\right) - \frac{\mu}{2}\int_0^T S_i d_i\right) \tag{42}$$

Where the maximum likelihood estimate of μ is

$$\widehat{\mu} = \frac{S(T)-S(0)}{\int_0^T S_i di} = \frac{S(T)-S(0)}{\sum_{i=0}^T S_i} \tag{43}$$

The approximation is a Riemann sum using the observed values $S_1, S_2, ..., S_T$. The approximate variance of $\widehat{\mu}$ is given by $\frac{\sigma^2}{\int_0^T S_i di}$. In order to estimate σ, let us consider $\ln S$, which is a diffusion with differential equation

$$\ln S(t) - \ln S(0) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW(t) \tag{44}$$

So that we can compute σ^2 from a path using the quadratic variation as

$$\sigma = \frac{1}{n}\sum_{i=0}^{2^n T} \ln^2\left[\frac{S(2^{-n} i)}{S(i-1)2^{-n}}\right] \tag{45}$$

The data are only observed to precision $n = 0$ yielding the approximate estimate.

$$\sigma^2 = \frac{1}{n}\sum_{i=0}^{T} \ln^2\left[\frac{S_i}{S_{i+1}}\right] \tag{46}$$

2.2.3 Determination of the market growth rate

The following result describes the growth rate of the market.

Theorem 1: For a twice continuously differentiable $F(S)$, i.e. $F \in C^{1,2}[(0,T),R]$, the solution of the time homogeneous investment equation

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2F(S)}{dS^2} + \mu S \frac{dF(S)}{dS} - r_t F(S) = -S \tag{47}$$

is given by

$$F(S) = \frac{A}{2}(1+S) + BS^{\lambda_1} + \frac{S}{(r_t - \mu)} \tag{48}$$

$$F(0) = \begin{cases} 0, & for\, A = 0 \\ \frac{A}{2}, & for\ A \neq 0 \end{cases} \tag{49}$$

and

$$\frac{dF(S)}{dS} = \frac{A}{2}(1+\hat{S}) + B\hat{S}^{\lambda_1} + \frac{\hat{S}}{(r_t - \mu)} = 0 \tag{50}$$

where A, B are constants. $\hat{S}$ is the expected equilibrium price of the primary security for a period t.

$$\left.\begin{aligned} \lambda_1 &= \frac{1}{2}\left[\left(1 - \frac{2\mu}{\sigma^2}\right) + \sqrt{\left(\frac{2\mu}{\sigma^2} - 1\right)^2 + \frac{8r_t}{\sigma^2}}\right] \\ \lambda_2 &= -\frac{1}{2}\left[\left(1 - \frac{2\mu}{\sigma^2}\right) + \sqrt{\left(\frac{2\mu}{\sigma^2} - 1\right)^2 + \frac{8r_t}{\sigma^2}}\right] \end{aligned}\right\} \tag{51}$$

λ_1 and λ_2 are positive and negative characteristic roots of (47) respectively.

Also $r_t = \mu + \sigma t,\ \mu > 0,\ \sigma \in [0,1]$, r_t is a decreasing (or increasing) linear function of time t as t increases. Now, let us denote

$$\frac{SdF}{dS} = F(S) \tag{52}$$

Differentiating (52), we have, $\frac{dF(S)}{dS} = \frac{dF}{dS} + S\frac{d^2F}{dS} \Rightarrow S\frac{d^2F}{dS} = \frac{dF}{dS} - \frac{dF}{dS} = 0$

Therefore,

$$S\frac{d^2F}{dS} = 0 \tag{53}$$

Substituting (52) and (53) into (47) then, we have that

$$\mu S\frac{dF(S)}{dS} - r_t\frac{dF(S)}{dS} = -S \tag{54}$$

Dividing through by S ,

$$\frac{dF(S)}{dS}(r_t - \mu) = 1 \Rightarrow \frac{dF(S)}{dS} = \frac{1}{(r_t - \mu)} = \eta$$

But, $r_t = \mu + \sigma_t$ thus

$$\eta = \frac{1}{\mu + \sigma t - \mu} = \frac{1}{\sigma t}$$

$$\eta = \frac{1}{\sigma t} \tag{55}$$

Equation (55) is called the market growth rate of the stock shares.

3 Applications and Numerical Example

We utilize the following data to interpret the above model equations. The data of United Bank for Africa (UBA) and Zenith Bank PLC quotations on the Nigerian Stock Exchange (NSE) between 1st November, 2011 and 1st January 2012 are presented in the Tables 1 and 2 below.

3.1 Fitting Analysis for the Model

(a) For Fitness of Table 1 (United Bank for Africa), we take t as a trading frequency and $T = 61$ days with $t = \frac{n}{T} \Rightarrow t = \frac{10}{61} = 0.164$

Table 1. United bank for Africa quotation on Nigerian stock exchange (NSE) market

S/No	Number of deals	Quantity	Value (N)	Share price (s)
1	266	22128090	56158178	2.55
2	155	51551481	118551842	2.26
3	166	81793301	177672927	2.17
4	231	43811777	107672927	2.50
5	197	83115894	227809283	2.60
6	272	47983738	153076719	3.16
7	209	37829415	113739208	2.99
8	244	22522995	66814563	3.00
9	207	10023772	28028523	2.77
10	250	48939528	156797970	3.25

Table 2. Zenith bank PLC quotation on Nigerian stock exchange (NSE) market

S/No	Number of Deals	Quantity	Value (N)	Share Price (s)
1	230	69798494	807375205	11.60
2	229	30348200	348413167	11.50
3	236	100076408	1150822455	11.50
4	207	4746149	57652153	11.30
5	180	44893463	529388520	11.99
6	215	65398291	842600794	12.90
7	301	38838291	489413638	12.60
8	316	31598675	394843225	12.40
9	315	16186423	197097419	12.30
10	406	92800309	1146994885	12.30

From (43), we have that

$$\hat{\mu} = \frac{S(T) - S(0)}{\int_0^T S_i di} = \frac{S(T) - S(0)}{\sum_{i=0}^{T} S_i}$$
$$= \frac{3.25 - 2.55}{27.5}$$
$$= 0.026$$

From (46),

$$\sigma^2 = \frac{1}{n}\sum_{i=0}^{T} \ln^2\left[\frac{S_i}{S_{i+1}}\right]$$
$$= \frac{1}{10}\sum_{i=0}^{T} \ln^2(1.103)$$
$$= \frac{0.0018}{10}$$
$$= 0.00018$$

Therefore, $\sigma = 0.0134$

We recall that;

$$\begin{aligned} r_t &= \mu + \sigma t \\ &= 0.026 + 0.0134(0.164) \\ &= 0.02813 \end{aligned}$$

Also from (51)

$$\begin{aligned} \lambda_1 &= \frac{1}{2}\left[\left(1-\frac{2\mu}{\sigma^2}\right)+\sqrt{\left(\frac{2\mu}{\sigma^2}-1\right)^2+\frac{8r_t}{\sigma^2}}\right] \\ &= \frac{1}{2}\left[\left(1-\frac{2(0.026)}{0.0134^2}\right)+\sqrt{\left(\frac{2(0.026)}{0.0134^2}-1\right)^2+\frac{8(0.02813)}{0.0134^2}}\right] \\ &= 1.0885 \end{aligned}$$

The market growth rate of the stock share is given by

$$\eta = \frac{1}{\sigma t} = \frac{1}{0.0134(0.164)} = 469.04$$

(b) For Fitness of Table 2 (Zenith Bank PLC), we take t as a trading frequencyand $T = 61$ days with $t = \frac{n}{T} \Rightarrow t = \frac{10}{61} = 0.164$

From (43), we have that

$$\begin{aligned} \hat{\mu} &= \frac{S(T)-S(0)}{\int_0^T S_i di} = \frac{S(T)-S(0)}{\sum_{i=0}^{T} S_i} \\ &= \frac{12.30-11.60}{120.39} \\ &= 0.0058 \end{aligned}$$

From (46),

$$\begin{aligned} \sigma^2 &= \frac{1}{n}\sum_{i=0}^{T}\ln^2\left[\frac{S_i}{S_{i+1}}\right] \\ &= \frac{1}{10}\sum_{i=0}^{T}\ln^2(1.107) \\ &= \frac{0.0019}{10} \\ &= 0.00019 \end{aligned}$$

Therefore, $\sigma = 0.0138$

We recall that;

$$\begin{aligned} r_t &= \mu + \sigma t \\ &= 0.026 + 0.0138(0.164) \\ &= 0.0081 \end{aligned}$$

Also from (51)

$$\begin{aligned} \lambda_1 &= \frac{1}{2}\left[\left(1 - \frac{2\mu}{\sigma^2}\right) + \sqrt{\left(\frac{2\mu}{\sigma^2} - 1\right)^2 + \frac{8r_t}{\sigma^2}}\right] \\ &= \frac{1}{2}\left[\left(1 - \frac{2(0.0058)}{0.0138^2}\right) + \sqrt{\left(\frac{2(0.0058)}{0.0138^2} - 1\right)^2 + \frac{8(0.0081)}{0.0138^2}}\right] \\ &= 1.3878 \end{aligned}$$

The market growth rate of the stock share is given by

$$\eta = \frac{1}{\sigma t} = \frac{1}{0.0138(0.164)} = 435.54$$

3.2 Numerical Example

We examine the effect of correlation coefficient, volatility and kurtosis on European options using the parameters below:

$$\sigma = 0.1, \rho = 0, \theta = 0.04, \kappa = 2, \nu_0 = 0.04, r = 0.01, S_0 = 1, Maturities: 0.5 - 3\,years$$

The effect of correlation coefficient on the skewness of the density function is shown in Fig. 1 with strike price *K = 0.092-0.11*, the effect of varying volatility on the kurtosis of the distribution is shown in Fig. 2 with strike price *K = 0.092-0.11*and the effect of changing the kurtosis of the distribution impacts on the implied volatility are shown in Figs. 3, 4 and 5 with strike price *K = 0.8 – 1.2*.The Figs. 1, 2, 3, 4 and 5 are shown in the appendix below.

3.3 Discussion of Results

From the Tables 1 and 2 above, we have the quotations of United Bank for Africa, Nigeria Plc. and Zenith Bank of Nigeria on NSE market respectively.The market growth rates of the stock share of United Bank for Africa, Nigeria Plc. and Zenith Bank of Nigeria between 1st November 2011 and 1st January 2012 were obtained to be 469.04 and 435.54 respectively.This result shows that the market growth rates of the stock share of United Bank for Africa, Nigeria Plc. is greater than that of its counterpart; Zenith Bank of Nigeria

Fig. 1 shows the effect of varying σ . The volatility, σ affects the kurtosis (peak) of the distribution. When $\sigma = 0$, the volatility is deterministic and the log-returns will be normally distributed. Increasing σ will then increase the kurtosis only, creating heavy tails on both sides.

Fig. 2 shows the effect of different values of correlation coefficient ρ on the skewness of the density function. The effect of changing the skewness of the distribution also impacts on the shape of the implied volatility surface. Also different values of ρ have effect on the implied volatility.

Figs. 3, 4 and 5 show the effect of changing the kurtosis of the distribution impacts on the implied volatility. The volatility σ affects the significance of the smile and skew. Increase in volatility makes the skew more prominent. This means that the market has a greater chance of extreme movements and that the volatility is more volatile.

4 Conclusion

Empirical studies have shown that an asset's log-return distribution is non-Gaussian. It is characterized by heavy tails and high peaks. There is also empirical evidence and economic arguments that suggest that equity returns and implied volatility are negatively correlated (also termed 'the leverage effect'). This departure from normality plagues the Black-Scholes with many problems. There are major differences between the stochastic volatility model and the well-known Black-Scholes model. The stochastic volatility model is superior to the Black-Scholes model. Theoretically, stochastic volatility models make more realistic assumptions; and empirically, researchers also find that the stochastic volatility model performs much better than the Black-Scholes model in pricing options. But stochastic volatility models are too technical and therefore not simple enough with respect to implementation. It is easy to understand and implement the Black-Scholes model. In contrast, Heston's model can imply a number of different distributions. The correlation parameter ρ can be interpreted as the relationship between the log-returns and volatility of the asset. Intuitively, if $\rho > 0$, then the volatility will increase as the asset price increases, this will spread the right tail and squeeze the left tail of distribution creating a fat right-tailed distribution. Conversely, if $\rho < 0$, then then the volatility will increase as the asset price decreases, this will spread the left tail and squeeze the right tail of distribution creating a fat left-tailed distribution (emphasizing the fact that equity returns and its related volatility are negatively correlated). Hence ρ , therefore, affect the skewness of the distribution. The effectsof implied volatility σ and the correlation ρ on the skewness, kurtosis on the density function are shown in the appendices below.

For many stochastic models, closed form solutions are not available. Some numerical methods are used. But usually it is time consuming to get the price using these numerical methods. Although Heston's stochastic option pricing model has a closed form solution, the infinite integral is still solved by a numerical method. It is much faster than other stochastic volatility models, it takes into account the leverage effect, its volatility updating structure permits analytical solutions to be generated for standard plain vanilla European options and thus the model allows a fast calibration to given market data. However, there remain some drawbacks such as; the integral needed for the computation of the option prices do not always converge fast enough. The standard Heston model usually fails to create a short term skew as strong as the one given by the market. In the real financial markets, prices exhibit jumps rather than continuous changes. Large price changes cannot be generated by pure diffusion processes in stochastic volatility models. [26] found that some parameters of Heston model need to be implausibly high when fitting the market data. One explanation for this is the absence of price jumps. The correlation parameter, ρ controls the level of skewness and the volatility of variance, σ controls the level of kurtosis. But the ability of Heston's model to generate enough short term kurtosis is limited. And a high level of σ means the short-term kurtosis is very high.

Modelling volatilities in a stochastic way corrects the simple constant volatility assumption of the Black-Scholes model. By changing the model parameters, almost all kinds of asset distributions can be generated by a negative correlation between the stock price process and the volatility process. The stochastic volatility model can also have an implied volatility surface which is similar to the one generated by market data. Among several stochastic volatility models, Heston stochastic model is the most popular.

Some further research may be done by incorporating dividend paying stocks in the dynamics of the underlying price process. Also to investigate the relationship between prices of participating contracts for which a separate hedge portfolio exists and those contracts for which it does not exist.

Competing Interests

Authors have declared that no competing interests exist.

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Appendices

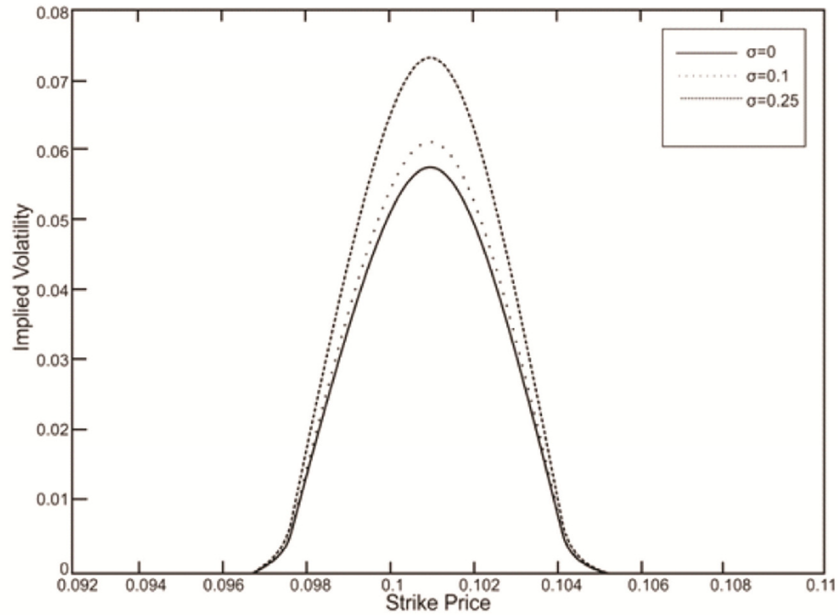

Fig. 1. The effect of σ on the kurtosis of the density function

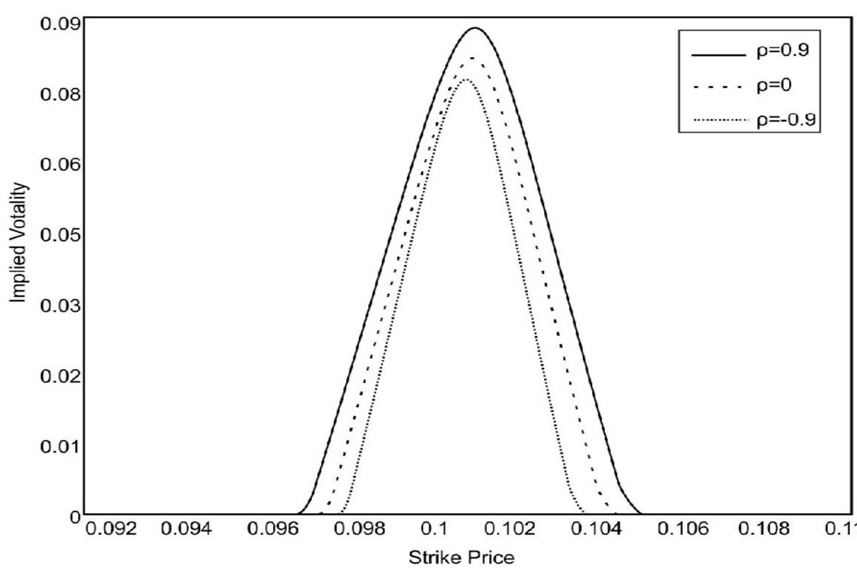

Fig. 2. The effect of ρ on the Skewness of the density function

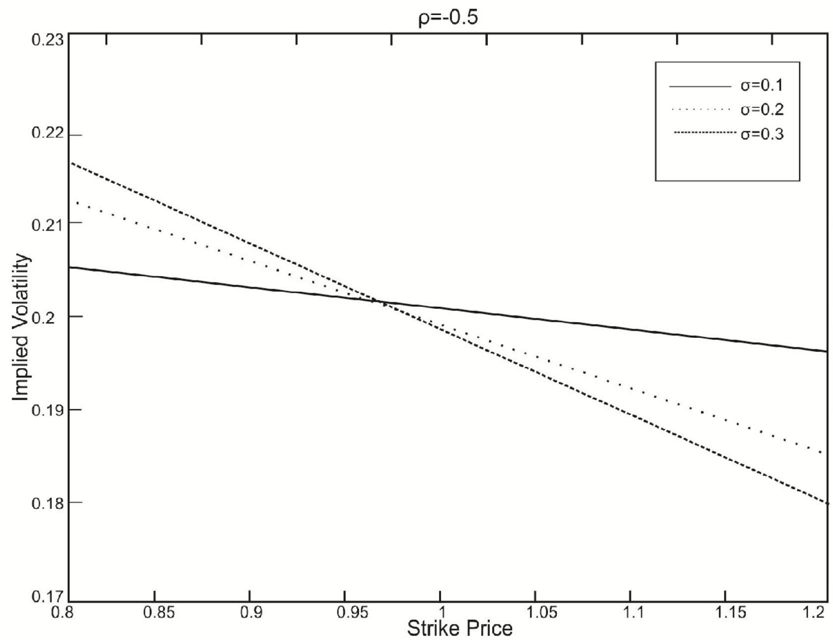

Fig. 3. Implied volatility surface when $\rho = -0.5$

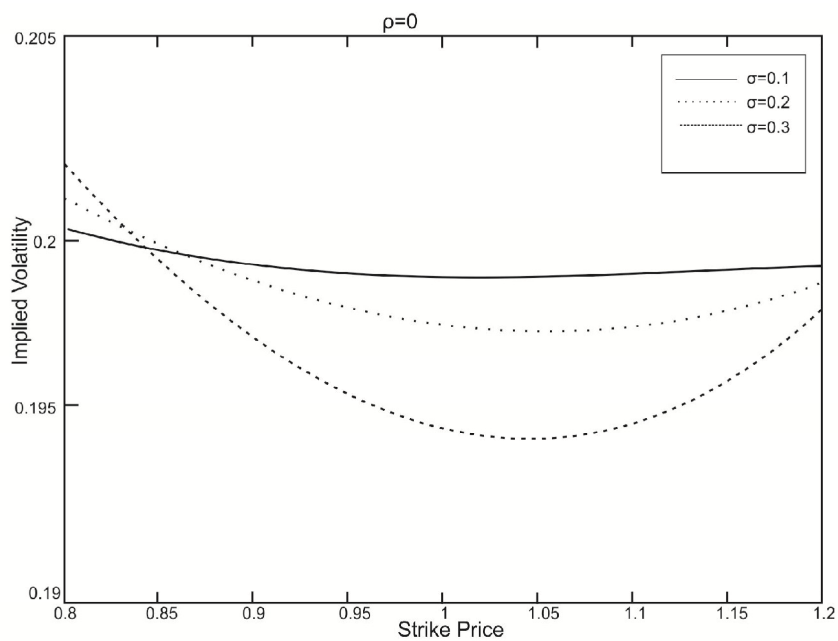

Fig. 4. Implied volatility surface when $\rho = 0$

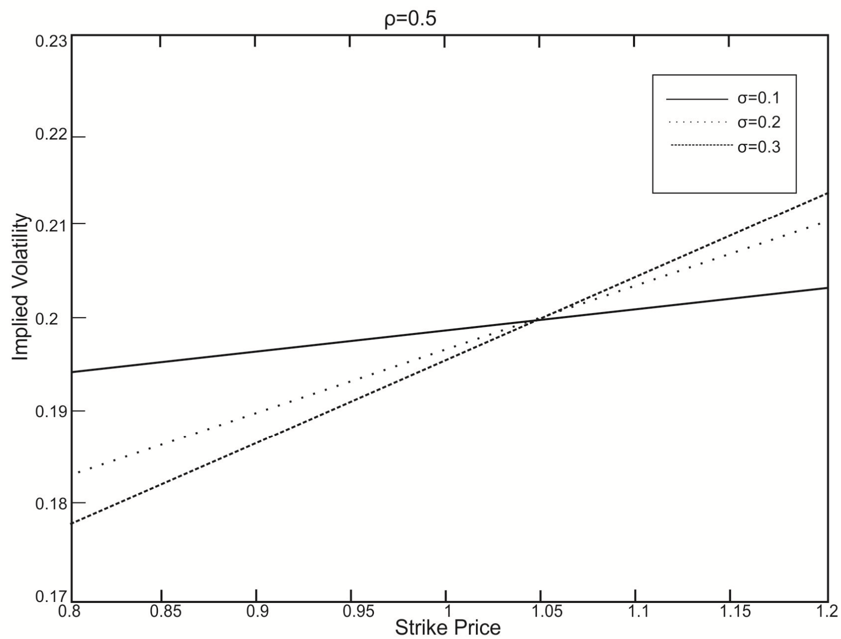

Fig. 5. Implied volatility surface when $\rho = 0.5$

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www.sciencedomain.org/review-history.php?iid=707&id=6&aid=6548