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Wigner's Theorem in s^* and $s_n(H)$ Spaces

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Abstract

Wigner theorem is the cornerstone of the mathematical formula of quantum mechanics, it has promoted the research of basic theory of quantum mechanics. In this article, we give a certain pair of functional equations between two real spaces s or two real spaces $s_n(H)$, that we called "phase isometry". It is obtained that all such solutions are phase equivalent to real linear isometries in the space s and the space $s_n(H)$.

Keywords

s Space, Wigner's Theorem, Phase Equivalent, Linear Isometry, $s_n(H)$ Space

1. Introduction

Mazur and Ulam in [1] proved that every surjective isometry U between X and Y is a affine, also states that the mapping with $U(0)=0$, then U is linear. Let X and Y be normed spaces, if the mapping $V: X \to Y$ satisfying that

$$\{\|V(x)-V(y)\|\} = \{\|x-y\|\} \quad (x, y \in X).$$

It was called isometry. About it's main properties in sequences spaces, Tingley, D, Ding Guanggui, Fu Xiaohong in [2] [3] [4] [5] [6] proved. So, we give a new definition that if there is a function $\varepsilon: X \to \{-1,1\}$ such that $J=\varepsilon V$ is a linear isometry. we can say the mapping $V: X \to Y$ is phase equivalent to J.

If the two spaces are Hilbert spaces, Rätz proved that the phase isometries $V: X \to Y$ are precisely the solutions of functional equation in [7]. If the two spaces are not inner product spaces, Huang and Tan [8] gave a partial answer about the real atomic L_p spaces with $p>0$. Jia and Tan [9] get the conclusion about the $\mathcal{L}$-type spaces. In [6], xiaohong Fu proved the problem of isometry extension in the s space detailedly.

In this artical, we mainly discuss that all mappings $V: s \to s$ or $s_n(H) \to s_n(H)$ also have the properties, that are solutions of the functional equation

$$\{\|V(x)-V(y)\|,\|V(x)+V(y)\|\}=\{\|x-y\|,\|x+y\|\}\quad (x,y\in X). \tag{1}$$

All metric spaces mentioned in this artical are assumed to be real.

2. Results about *s*

First, let us introduction some concepts. The *s* space in [10], which consists of all scalar sequences and for each elements $x=\{\xi_k\}=\sum_k \xi_k e_k$, the F-norm of x is defined by $\|x\|=\sum_{n=1}^{\infty}\frac{1}{2^k}\frac{|\xi_k|}{1+|\xi_k|}$. Let $s_{(n)}$ denote the set of all elements of the form $x=\{\xi_1,\cdots,\xi_n\}$ with $\|x\|=\sum_{k=1}^{n}\frac{1}{2^k}\frac{|\xi_k|}{1+|\xi_k|}$. where $e_k=\{\xi_{k'}:\xi_k=1,\xi_{k'}=0,k'\neq k,\text{for all } k'\in\Gamma\}$. We denote the support of x by Γ_x, *i.e.*,

$$supp(x)=\Gamma_x=\{\gamma\in\Gamma:\xi_\gamma\neq 0\}.$$

For all $x,y\in s$, if $\Gamma_x\cap\Gamma_y=\varnothing$, we say that x is orthogonal to y and write $x\perp y$.

Lemma 2.1. Let $S_{r_0}(s)$ be a sphere with radius r_0 and center 0 in *s*. Suppose that $V_0:S_{r_0}(s)\to S_{r_0}(s)$ is a mapping satisfying Equation (1). Then for any $x,y\in S_{r_0}(s)$, we have

$$x\perp y\Leftrightarrow V_0(x)\perp V_0(y)$$

Proof: Necessity. Choosing $\forall x=\{\xi_n\}$, $y=\{\eta_n\}\in S_{r_0}(s)$ that satisfying $x\perp y$. We can suppose $V_0(x)=\{\xi'_n\}$, $V_0(y)=\{\eta'_n\}$. And we also have

$$\{\|V_0(x)-V_0(y)\|,\|V_0(x)+V_0(y)\|\}=\{\|x-y\|,\|x+y\|\}.$$

So

$$\|V_0(x)-V_0(y)\|=\|x-y\|=\|x\|+\|y\|=2r_0=\|V_0(x)\|+\|V_0(y)\|$$

or

$$\|V_0(x)-V_0(y)\|=\|x+y\|=\|x\|+\|y\|=2r_0=\|V_0(x)\|+\|V_0(y)\|$$

Thus

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|\xi'_n-\eta'_n|}{1+|\xi'_n-\eta'_n|}=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|\xi'_n|}{1+|\xi'_n|}+\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|\eta'_n|}{1+|\eta'_n|}$$

That means

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\left[\frac{|\xi'_n-\eta'_n|}{1+|\xi'_n-\eta'_n|}-\frac{|\xi'_n|}{1+|\xi'_n|}-\frac{|\eta'_n|}{1+|\eta'_n|}\right]=0 \tag{2}$$

It is easy to know $f(x)=\frac{x}{1+x}$ is strictly increasing. And $|\xi'_n-\eta'_n|\le|\xi'_n|+|\eta'_n|$. We can get the result $\xi'_n\cdot\eta'_n=0$.

For $\|V_0(x)+V_0(y)\|$, similarty to the above $(|\xi_n'+\eta_n'|\le|\xi_n'|+|\eta_n'|)$. It is $V_0(x)\perp V_0(y)$. Sufficiency. For $V_0(x)\perp V_0(y)$, that is, $\xi_n'\cdot\eta_n'=0$, so (2) holds, and we have

$$\|x-y\|=\|V_0(x)-V_0(y)\|=\|V_0(x)\|+\|V_0(y)\|=2r_0$$

so, it must have $\|x-y\|=\|x\|+\|y\|$.

or

$$\|x-y\|=\|V_0(x)+V_0(y)\|=\|V_0(x)\|+\|V_0(y)\|=2r_0$$

as the same $\|x-y\|=\|x\|+\|y\|$. It follows that

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|\xi_n-\eta_n|}{1+|\xi_n-\eta_n|}=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|\xi_n|}{1+|\xi_n|}+\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|\eta_n|}{1+|\eta_n|} \tag{3}$$

Similarly to the proof of necessity, we get $x\perp y$.

Lemma 2.2. Let $S_{r_0}(s_{(n)})$ be a sphere with radius r_0 in the finite dimensional space $s_{(n)}$, where $r_0<\frac{1}{2^n}$. Suppose that $V_0:S_{r_0}(s_{(n)})\to S_{r_0}(s_{(n)})$ is an phase isometry. Let $\lambda_k=\frac{2^k r_0}{1-2^k r_0}(k\in\mathbb{N},1\le k\le n)$, then there is a unique real θ with $|\theta|=1$, such that $V_0(\lambda_k e_k)=\theta\lambda_k e_k$.

Proof: We proof first that for any $k(1\le k\le n)$, there is a unique $l(1\le l\le n)$ and a unique real θ with $|\theta|=1$ such that $V_0(\lambda_k e_k)=\theta\lambda_l e_l$ (because the assumption of λ_k implies $\lambda_k e_k\in S_{r_0}(s_{(n)})$). To this end, suppose on the contrary that $V_0(\lambda_{k_0}e_{k_0})=\sum_{k=1}^{n}\eta_k e_k$ and $\eta_{k_1}\ne0,\eta_{k_2}\ne0$. In view of Lemma 1, we have

$$[suppV_0(\lambda_{k_0}e_{k_0})]\cap[suppV_0(\lambda_k e_k)]=\varnothing\quad\forall k\ne k_0,1\le k\le n.$$

Hence, by the "pigeon nest principle" (or Pigeonhole principle) there must exist $k_{i_0}(1\le k_{i_0}\le n)$ such that $V_0(\lambda_{k_{i0}}e_{k_{i0}})=\theta$, which leads to a contradiction.

Next, if $V_0(\lambda_k e_k)=\theta_1\lambda_l e_l$, $V_0(-\lambda_k e_k)=\theta_2\lambda_p e_p$, where $|\theta_1|=|\theta_2|=1$, then $l=p$ and $\theta_2=-\theta_1$. Indeed, if $l\ne p$, we have

$$\|V(\lambda_k e_k)-V(-\lambda_k e_k)\|=\|2\lambda_k e_k\|=\frac{1}{2^k}\frac{|2\lambda_k|}{1+|2\lambda_k|}\ne2r_0$$

or

$$\|V(\lambda_k e_k)-V(-\lambda_k e_k)\|=0$$

and

$$\|V(\lambda_k e_k)-V(-\lambda_k e_k)\|=\|\theta_1\lambda_l e_l-\theta_2\lambda_p e_p\|=2r_0 \tag{4}$$

a contradiction which implies $l=p$. From this $\theta_1=-\theta_2$ follows. Finally, there is a unique θ with $|\theta|=1$ such that $V_0(\lambda_k e_k)=\theta\lambda_k e_k$. Indeed, if $V_0(\lambda_k e_k)=\theta\lambda_l e_l$, by the result in the last step, we have $V_0(-\lambda_k e_k)=-\theta\lambda_l e_l$, thus

$$\left\{\left\|V\left(\lambda_k e_k\right)+V\left(-\lambda_k e_k\right)\right\|,\left\|V\left(\lambda_k e_k\right)-V\left(-\lambda_k e_k\right)\right\|\right\}$$
$$=\left\{\left\|2\lambda_k e_k\right\|,0\right\}=\left\{\frac{1}{2^k}\frac{\left|2\lambda_k\right|}{1+\left|2\lambda_k\right|},0\right\}$$

and

$$\left\{\left\|V\left(\lambda_k e_k\right)+V\left(-\lambda_k e_k\right)\right\|,\left\|V\left(\lambda_k e_k\right)-V\left(-\lambda_k e_k\right)\right\|\right\}$$
$$=\left\{\left\|2\theta\lambda_l e_l\right\|,0\right\}=\left\{\frac{1}{2^l}\frac{\left|2\lambda_l\right|}{1+\left|2\lambda_l\right|},0\right\} \tag{5}$$

So, we get

$$\frac{1}{2^k}\frac{\left|2\lambda_k\right|}{1+\left|2\lambda_k\right|}=\frac{1}{2^l}\frac{\left|2\lambda_l\right|}{1+\left|2\lambda_l\right|}$$

and we also have

$$\frac{1}{2^k}\frac{\left|\lambda_k\right|}{1+\left|\lambda_k\right|}=\frac{1}{2^l}\frac{\left|\lambda_l\right|}{1+\left|\lambda_l\right|},(=r_0)$$

through the two equalities of above

$$\frac{\frac{1}{2^k}\frac{\left|2\lambda_k\right|}{1+\left|2\lambda_k\right|}}{\frac{1}{2^k}\frac{\left|\lambda_k\right|}{1+\left|\lambda_k\right|}}=\frac{\frac{1}{2^l}\frac{\left|2\lambda_l\right|}{1+\left|2\lambda_l\right|}}{\frac{1}{2^l}\frac{\left|\lambda_l\right|}{1+\left|\lambda_l\right|}}$$

In the end,

$$\left|\lambda_l\right|=\left|\lambda_k\right| \tag{6}$$

The proof is complete.

Lemma 2.3. Let $X=S_{r_0}\left(s_{(n)}\right)$ and $Y=S_{r_0}\left(s_{(n)}\right)$. Suppose that $V_0:X\to Y$ is a surjective mapping satisfying Equation (1) and λ_k as in Lemma 2.2. Then for any lement $x=\sum_k \xi_k e_k\in X$, we have $V_0(x)=\sum_k \eta_k e_k$, where $\left|\xi_k\right|=\left|\eta_k\right|$ for any $1\le k_0\le n$.

Proof: Note that the defination of V_0, we can easily get $V_0(0)=0$. For any $0\neq x\in X$, write $x=\sum_k \xi_k e_k$, where $\sum_k\frac{1}{2^k}\frac{\left|\xi_k\right|}{1+\left|\xi_k\right|}=r_0$. we can write $V_0(x)=\sum_k \eta_k e_k$, where $\sum_k\frac{1}{2^k}\frac{\left|\eta_k\right|}{1+\left|\eta_k\right|}=r_0$. we have

$$\left\|V_0(x)+V_0\left(\lambda_{k_0}e_{k_0}\right)\right\|+\left\|V_0(x)-V_0\left(\lambda_{k_0}e_{k_0}\right)\right\|$$
$$=\left\|x+\lambda_{k_0}e_{k_0}\right\|+\left\|x-\lambda_{k_0}e_{k_0}\right\|$$
$$=\left\|\sum_{k\neq k_0}\xi_k e_k+\left(\xi_{k_0}+\lambda_{k_0}\right)e_{k_0}\right\|+\left\|\sum_{k\neq k_0}\xi_k e_k+\left(\xi_{k_0}-\lambda_{k_0}\right)e_{k_0}\right\|$$
$$=r_0+\frac{1}{2^{k_0}}\frac{\left|\xi_{k_0}+\lambda_{k_0}\right|}{1+\xi_{k_0}+\lambda_{k_0}}-\frac{1}{2^{k_0}}\frac{\left|\xi_{k_0}\right|}{1+\left|\xi_{k_0}\right|}+r_0+\frac{1}{2^{k_0}}\frac{\left|\xi_{k_0}-\lambda_{k_0}\right|}{1+\xi_{k_0}-\lambda_{k_0}}-\frac{1}{2^{k_0}}\frac{\left|\xi_{k_0}\right|}{1+\left|\xi_{k_0}\right|}.$$

On the other hand, we have

$$\begin{aligned}&\left\|V_0(x)+V_0\left(\lambda_{k_0}e_{k_0}\right)\right\|+\left\|V_0(x)-V_0\left(\lambda_{k_0}e_{k_0}\right)\right\|\\&=\left\|\sum_{k=1}^{n}\eta_k e_k+\theta_{k_0}\lambda_{k_0}e_{k_0}\right\|+\left\|\sum_{k=1}^{n}\eta_k e_k-\theta_{k_0}\lambda_{k_0}e_{k_0}\right\|\\&=\left\|\sum_{k\neq k_0}\eta_k e_k+\left(\eta_{k_0}+\theta_{k_0}\lambda_{k_0}\right)e_{k_0}\right\|+\left\|\sum_{k\neq k_0}\eta_k e_k+\left(\eta_{k_0}-\theta_{k_0}\lambda_{k_0}\right)e_{k_0}\right\|\\&=r_0+\frac{1}{2^{k_0}}\frac{\left|\eta_{k_0}+\theta_{k_0}\lambda_{k_0}\right|}{1+\left|\eta_{k_0}+\theta_{k_0}\lambda_{k_0}\right|}-\frac{1}{2^{k_0}}\frac{\left|\eta_{k_0}\right|}{1+\left|\eta_{k_0}\right|}+r_0+\frac{1}{2^{k_0}}\frac{\left|\eta_{k_0}-\theta_{k_0}\lambda_{k_0}\right|}{1+\left|\eta_{k_0}-\theta_{k_0}\lambda_{k_0}\right|}-\frac{1}{2^{k_0}}\frac{\left|\eta_{k_0}\right|}{1+\left|\eta_{k_0}\right|}.\end{aligned}$$

Combiniing the two equations, we obtain that

$$\begin{aligned}&\frac{\left|\xi_{k_0}+\lambda_{k_0}\right|}{1+\left|\xi_{k_0}+\lambda_{k_0}\right|}-\frac{\left|2\xi_{k_0}\right|}{1+\left|\xi_{k_0}\right|}+\frac{\left|\xi_{k_0}-\lambda_{k_0}\right|}{1+\left|\xi_{k_0}-\lambda_{k_0}\right|}\\&=\frac{\left|\eta_{k_0}+\theta_{k_0}\lambda_{k_0}\right|}{1+\left|\eta_{k_0}+\theta_{k_0}\lambda_{k_0}\right|}-\frac{\left|2\eta_{k_0}\right|}{1+\left|\eta_{k_0}\right|}+\frac{\left|\eta_{k_0}-\theta_{k_0}\lambda_{k_0}\right|}{1+\left|\eta_{k_0}-\theta_{k_0}\lambda_{k_0}\right|}\end{aligned}$$

As $\lambda_{k_0}\geq\left|\xi_{k_0}\right|$ and $\lambda_{k_0}\geq\left|\eta_{k_0}\right|$, we have

$$\begin{aligned}&\frac{\xi_{k_0}+\lambda_{k_0}}{1+\xi_{k_0}+\lambda_{k_0}}-\frac{2\xi_{k_0}}{1+\xi_{k_0}}+\frac{\lambda_{k_0}-\xi_{k_0}}{1+\lambda_{k_0}-\xi_{k_0}}\\&=\frac{\lambda_{k_0}+\theta_{k_0}\eta_{k_0}}{1+\lambda_{k_0}+\theta_{k_0}\eta_{k_0}}-\frac{2\eta_{k_0}}{1+\eta_{k_0}}+\frac{\lambda_{k_0}-\theta_{k_0}\eta_{k_0}}{1+\lambda_{k_0}-\theta_{k_0}\eta_{k_0}}\end{aligned}$$

Therefore,

$$\frac{\lambda_{k_0}+\lambda_{k_0}^2-\xi_{k_0}^2}{\left(1+\lambda_{k_0}\right)^2-\xi_{k_0}^2}+\frac{\eta_{k_0}}{1+\eta_{k_0}}-\frac{\xi_{k_0}}{1+\xi_{k_0}}=\frac{\lambda_{k_0}+\lambda_{k_0}^2-\eta_{k_0}^2}{\left(1+\lambda_{k_0}\right)^2-\eta_{k_0}^2}$$

Analysis of the equation, according to the monotony of the function, that is

$$\left|\xi_k\right|=\left|\eta_k\right| \tag{7}$$

The proof is complete.□

The next result shows that a mapping satisfying functional Equation (1) has a property close to linearity.

Lemma 2.4. Let $X=s_{(n)}$ and $Y=s_{(n)}$. Suppose that $V:X\to Y$ is a surjective mapping satisfying Equation (1). there exist two real numbers α and β with absolute 1 such that

$$V(x+y)=\alpha V(x)+\beta V(y)$$

for all nonzero vectors x and y in X, x and y are orthogonal.

Proof: Let x and y be nonzero orthogonal vectors in X, we write $x=\sum_k\xi_k e_k$, $y=\sum_k\eta_k e_k$.

$$V(x)=\sum_k\xi_k' e_k,\quad V(y)=\sum_k\eta_k' e_k$$

$$V(x+y)=\sum_k\xi_k'' e_k+\sum_k\eta_k'' e_k,$$

where $\left|\xi_k'\right|=\left|\xi_k''\right|=\left|\xi_k\right|$ and $\left|\eta_k'\right|=\left|\eta_k''\right|=\left|\eta_k\right|$. We infer from Equation (1) that

$$\begin{aligned}
&\{\|2x\|+\|y\|,\|y\|\}\\
&=\{\|V(x+y)+V(x)\|,\|V(x+y)-V(x)\|\}\\
&=\left\{\left\|\sum_k \xi_k'' e_k+\sum_k \eta_k'' e_k+\sum_k \xi_k' e_k\right\|,\left\|\sum_k \xi_k'' e_k+\sum_k \eta_k'' e_k+\sum_k \eta_k' e_k\right\|\right\}\\
&=\left\{\frac{1}{2^k}\frac{|\xi_k''+\xi_k'|}{1+|\xi_k''+\xi_k'|}+\|y\|,\frac{1}{2^k}\frac{|\xi_k''-\xi_k'|}{1+|\xi_k''-\xi_k'|}+\|y\|\right\}
\end{aligned}$$

Through the above equation we can get $\xi_k''+\xi_k'=0$ or $\xi_k''-\xi_k'=0$. This implies that $\sum_k \xi_k'' e_k=\pm V(x)$, and similarly $\sum_k \eta_k'' e_k=\pm V(y)$. The proof is complete.□

Lemma 2.5. Let $X=s$ and $Y=s$. Suppose that $V:X\to Y$ is a surjective mapping satisfying Equation (1). Then V is injective and $V(-x)=-V(x)$ for all $x\in X$.

Proof: Suppose that V is surjective and $V(x)=V(y)$ for some $x,y\in X$. Putting $y=x$ in the Equation (1), this yields

$$\{\|2V(x)\|,0\}=\{\|2x\|,0\}$$

$V(x)=0$ if and only if $x=0$. Assume that $V(x)=V(y)\neq 0$ choose $z\in X$ such that $V(z)=-V(x)$, using the Equation (1) for x,y,z, we obtain

$$\{\|x-y\|,\|x+y\|\}=\{\|V(x)+V(y)\|,\|V(x)-V(y)\|\}=\{\|2V(x)\|,0\}$$

$$\{\|x-z\|,\|x+z\|\}=\{\|V(x)+V(z)\|,\|V(x)-V(z)\|\}=\{\|2V(x)\|,0\}$$

This yields $y,z\in\{x,-x\}$. If $z=x$, then $V(x)=-V(x)=0$, which is a contradiction. So we obtain $z=-x$, and we must have $y=x$. For otherwise we get $y=z=-x$ and

$$V(x)=V(y)=V(z)=-V(x)=0$$

This lead to the contradiction that $V(x)\neq 0$.

Theorem 2.6. Let $X=s_{(n)}$ and $Y=s_{(n)}$. Suppose that $V:X\to Y$ is a surjective mapping satisfying Equation (1). Then V is phase equivalent to a linear isometry *J*.

Proof: Fix $\gamma_0\in\Gamma$, and let $Z=\{z\in X:z\perp e_{\gamma_0}\}$. By Lemma 2.4 we can write

$$V(z+\lambda e_{\gamma_0})=\alpha(z,\lambda)V(z)+\beta(z,\lambda)V(\lambda e_{\gamma_0}),|\alpha(z,\lambda)|=|\beta(z,\lambda)|=1$$

for any $z\in Z$. Then, we can define a mapping $J:s_{(n)}\to s_{(n)}$ as follows:

$$J(z+\lambda e_{\gamma_0})=\alpha(z,\lambda)\beta(z,\lambda)V(z)+V(\lambda e_{\gamma_0})$$

$$J(\lambda z)=\alpha(z,\lambda)\beta(z,\lambda)V(\lambda z)$$

$$J(e_{\gamma_0})=V(e_{\gamma_0}),\ J(-e_{\gamma_0})=-V(e_{\gamma_0})$$

for $\forall 0\neq\lambda\in\mathbb{R}$. The *J* is phase equivalent to *V*. So it is easily to know that *J* satisfies functional Equation (1). For any $z\in Z$, and $\forall 0\neq\lambda\in\mathbb{R}$,

$$\left\{\|2z\|+\frac{1}{2^{\gamma_0}}\frac{|1+\lambda|}{1+|1+\lambda|},\frac{1}{2^{\gamma_0}}\frac{|1-\lambda|}{1+|1-\lambda|}\right\}$$
$$=\left\{\left\|J\left(z+e_{\gamma_0}\right)+J\left(z+\lambda e_{\gamma_0}\right)\right\|,\left\|J\left(z+e_{\gamma_0}\right)-J\left(z+\lambda e_{\gamma_0}\right)\right\|\right\}$$
$$=\left\{\left\|\alpha(z,1)\beta(z,1)V(z)+\alpha(z,\lambda)\beta(z,\lambda)V(z)+V\left(e_{\gamma_0}\right)+V\left(\lambda e_{\gamma_0}\right)\right\|,\right.$$
$$\left.\left\|\alpha(z,1)\beta(z,1)V(z)-\alpha(z,\lambda)\beta(z,\lambda)V(z)+V\left(e_{\gamma_0}\right)-V\left(\lambda e_{\gamma_0}\right)\right\|\right\}$$
$$=\left\{\left\|\left|\alpha(z,1)\beta(z,1)+\alpha(z,\lambda)\beta(z,\lambda)\right|V(z)\right\|+\frac{1}{2^{\gamma_0}}\frac{|1+\lambda|}{1+|1+\lambda|},\right.$$
$$\left.\left\|\left|\alpha(z,1)\beta(z,1)-\alpha(z,\lambda)\beta(z,\lambda)\right|V(z)\right\|+\frac{1}{2^{\gamma_0}}\frac{|1+\lambda|}{1+|1+\lambda|}\right\}$$

That means $\alpha(z,1)\beta(z,1)=\alpha(z,\lambda)\beta(z,\lambda)$, $J\left(z+\lambda e_{\gamma_0}\right)=J(z)+V\left(\lambda e_{\gamma_0}\right)$ for any $z\in Z$, and $\forall 0\neq\lambda\in\mathbb{R}$. That yields

$$\left\{\left\|J(z)+J(-z)\right\|,\left\|J(z)-J(-z)+2V\left(e_{\gamma_0}\right)\right\|\right\}$$
$$=\left\{\left\|J\left(z+e_{\gamma_0}\right)+J\left(-z-e_{\gamma_0}\right)\right\|,\left\|J\left(z+e_{\gamma_0}\right)-J\left(-z-e_{\gamma_0}\right)\right\|\right\}$$
$$=\left\{0,\left\|2\left(z+e_{\gamma_0}\right)\right\|\right\}$$

That means $J(-z)=-J(z)$. On the other hand,

$$\left\{\|z_1+z_2\|+\frac{2}{3}\frac{1}{2^{\gamma_0}},\|z_1-z_2\|\right\}$$
$$=\left\{\left\|J\left(z_1+e_{\gamma_0}\right)+J\left(z_2+e_{\gamma_0}\right)\right\|,\left\|J\left(z_1+e_{\gamma_0}\right)-J\left(z_2+e_{\gamma_0}\right)\right\|\right\}$$
$$=\left\{\left\|J(z_1)+J(z_2)\right\|+\frac{2}{3}\frac{1}{2^{\gamma_0}},\left\|J(z_1)-J(z_2)\right\|\right\}$$

for $\forall z_1,z_2\in Z$, It follows that $\|J(x)-J(y)\|=\|x-y\|$ for all $x,y\in X$, by assumed conditions, so *J* is a surjective isometry.□

Theorem 2.7. Let $X=s$ and $Y=s$. Suppose that $V:X\to Y$ is a surjective mapping satisfying Equation (1). Then V is phase equivalent to a linear isometry *J*.

Proof: According to [10] Theorem 1, Theorem 2 the author presents some results of extension from some spheres in the finite dimensional spaces $s_{(n)}$. And also we have the above Theorem 2.6, so we can get the result easily.

3. Results about $s_n(H)$

In this part, we mainly introduce the space $s_n(H)$, where *H* is a Hilbert space. In [11] mainly discussed the isometric extension in the space $s_n(H)$. For each element $x=\{x(k)\}$, the F-norm of x is defined by $\|x\|=\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{\|x(k)\|}{1+\|x(k)\|}$. Let $s_n(H)$ denote the set of all elements of the form $x=(x(1),\cdots,x(n))$ with

$\|x\| = \sum_{k=1}^{n} \frac{1}{2^k} \frac{\|x(k)\|}{1+\|x(k)\|}$. where $x(i)(i=1,\cdots,n) \in H$.

Some notations used:

$$e_{x(k)} = (0,\cdots,x(k),\cdots,0) \in s_n(H), \text{ where } \|x(k)\| = 1.$$

Specially, when $\|x(k)\| = 0$, we have $e_{\frac{x(k)}{\|x(k)\|}} = (0,\cdots,0)$.

Next, we study the phase isometry between the space $s_n(H)$ to $s_n(H)$, that if V is a surjective phase isometry, then V is phase equivalent to a linear isometry *J*.

Lemma 3.1. If $x, y \in s_n(H)$, then

$$\|x-y\| = \|x\| + \|y\| \text{ if and only if } suppx \cap suppy = \varnothing$$

where $suppx = \{n : x(n) \neq 0, n \in \mathbb{N}\}$.

Proof: It has a detailed proof process in [11].

Lemma 3.2. Let $S_{r_0}(s_n(H))$ be a sphere with radius r_0 in the finite dimensional space $s_n(H)$, where $r_0 < \frac{1}{2^n}$. Defined

$V_0 : S_{r_0}(s_n(H)) \to S_{r_0}(s_n(H))$ is an phase isometry, then we can get

$$x \perp y \Leftrightarrow V_0(x) \perp V_0(y).$$

Proof: "⇒" Take any two elements $x = \{x(i)\}$, $y = \{y(i)\}$, let $V_0(x) = \{x'(i)\}$, $V_0(y) = \{y'(i)\}$. Then we have

$$2r_0 = \|x\| + \|y\| = \|x-y\| = \|V_0(x) - V_0(y)\| = \sum_{i=1}^{n} \frac{1}{2^i} \frac{\|x'(i) - y'(i)\|}{1+\|x'(i) - y'(i)\|}$$

or

$$2r_0 = \|x\| + \|y\| = \|x-y\| = \|V_0(x) + V_0(y)\| = \sum_{i=1}^{n} \frac{1}{2^i} \frac{\|x'(i) - y'(i)\|}{1+\|x'(i) - y'(i)\|} \tag{8}$$

at the same time, we have

$$\sum_{i=1}^{n} \frac{1}{2^i} \frac{\|x'(i) - y'(i)\|}{1+\|x'(i) - y'(i)\|} \le \sum_{i=1}^{n} \frac{1}{2^i} \frac{\|x'(i)\|}{1+\|x'(i)\|} + \sum_{i=1}^{n} \frac{1}{2^i} \frac{\|y'(i)\|}{1+\|y'(i)\|} = 2r_0$$

$$\sum_{i=1}^{n} \frac{1}{2^i} \frac{\|x'(i) + y'(i)\|}{1+\|x'(i) + y'(i)\|} \le \sum_{i=1}^{n} \frac{1}{2^i} \frac{\|x'(i)\|}{1+\|x'(i)\|} + \sum_{i=1}^{n} \frac{1}{2^i} \frac{\|y'(i)\|}{1+\|y'(i)\|} = 2r_0 \tag{9}$$

That means $\|V_0(x) - V_0(y)\| = \|V_0(x) + V_0(y)\| = \|V_0(x)\| + \|+V_0(y)\|$, it is $V_0(x) \perp V_0(y)$. "⇐" The proof of sufficiency is similar to the Lemma 2.1.

Lemma 3.3. Let V_0 be as in Lemma 3.2, $\lambda_k = \frac{2^k r_0}{1-2^k r_0}(k \in \mathbb{N}), (1 \le k \le n)$, and $e_{x(k)} = s_n(H)$. $(\|x(k)\| = 1)$. Then there exists $x'(k) \in H(\|x'(k)\| = 1)$, such that $V_0(\pm\lambda_k e_{x(k)}) = \pm\lambda_k e_{x'(k)}$.

Proof: We prove first that, for any $k(1 \le k \le n)$, there exist $l(1 \le l \le n)$ and

$x'(l)\left(\|x'(l)\|=1\right)$ such that $V_0\left(\lambda_k e_{x(k)}\right)=\lambda_l e_{x'(k)}$. And then prove $l=p$. It is the same an Lemma 2.2.

Finally, we assert that, there exists $x'(k)$ such that $V_0\left(\pm\lambda_k e_{x(k)}\right)=\pm\lambda_k e_{x'(k)}$. Indeed, if $V_0\left(\lambda_k e_{x(k)}\right)=\lambda_l e_{x'(l)}$, by the result in the last step, we have

$V_0\left(-\lambda_k e_{x(k)}\right)=\lambda_l e_{x''(l)}$,

$$\begin{aligned}&\left\{0,\frac{1}{2^k}\frac{2\lambda_k}{1+2\lambda_k}\right\}\\&=\left\{\left\|V_0\left(\lambda_k e_{x(k)}\right)-V_0\left(-\lambda_k e_{x(k)}\right)\right\|,\left\|V_0\left(\lambda_k e_{x(k)}\right)+V_0\left(-\lambda_k e_{x(k)}\right)\right\|\right\}\\&=\left\{\left\|\lambda_l e_{x'(l)}-\lambda_l e_{x''(l)}\right\|,\left\|\lambda_l e_{x'(l)}-\lambda_l e_{x''(l)}\right\|\right\}\\&=\left\{\frac{1}{2^l}\frac{\lambda_l\|x'(l)-x''(l)\|}{1+\lambda_l\|x'(l)-x''(l)\|},\frac{1}{2^l}\frac{\lambda_l\|x'(l)+x''(l)\|}{1+\lambda_l\|x'(l)+x''(l)\|}\right\}\end{aligned}$$

Therefore,

$$\frac{1}{2^k}\frac{2\lambda_k}{1+2\lambda_k}=\frac{1}{2^l}\frac{\lambda_l\|x'(l)-x''(l)\|}{1+\lambda_l\|x'(l)-x''(l)\|}\le\frac{1}{2^l}\frac{2\lambda_l}{1+2\lambda_l}$$

or

$$\frac{1}{2^k}\frac{2\lambda_k}{1+2\lambda_k}=\frac{1}{2^l}\frac{\lambda_l\|x'(l)+x''(l)\|}{1+\lambda_l\|x'(l)+x''(l)\|}\le\frac{1}{2^l}\frac{2\lambda_l}{1+2\lambda_l}\tag{10}$$

So, we can get $k=l$. And $\|x'(l)-x''(l)\|=\|x'(l)+x''(l)\|=2$, that means $x'(l)=\pm x''(l)$.

Lemma 3.4. Let $X=s_n(H)$ and $Y=s_n(H)$. Suppose that $V:X\to Y$ is a surjective mapping satisfying Equation (1). there exist two real numbers α and β with absolute 1 such that

$$V(x+y)=\alpha V(x)+\beta V(y)$$

for all nonzero vectors *x* and *y* in *X*, *x* and *y* are orthogonal. **Proof:** Let $x=\{x(i)\}$ and $y=\{y(i)\}$ be nonzero orthogonal vectors in *X*.

$$V\{x(i)\}=\sum_{i=1}^{n}\frac{\|x(i)\|}{\lambda_i}V\left(\lambda_i e_{\frac{x(i)}{\|x(i)\|}}\right),$$

$$V\{y(i)\}=\sum_{i=1}^{n}\frac{\|y(i)\|}{\mu_i}V\left(\mu_i e_{\frac{y(i)}{\|y(i)\|}}\right)$$

$$V\{x(i)+y(i)\}=\sum_{i=1}^{n}\frac{\|x'(i)\|}{\lambda_i}V\left(\lambda_i e_{\frac{x'(i)}{\|x'(i)\|}}\right)+\sum_{i=1}^{n}\frac{\|y'(i)\|}{\mu_i}V\left(\mu_i e_{\frac{y'(i)}{\|y'(i)\|}}\right),$$

where $\|x'(i)\|=\|x(i)\|$ and $\|y'(i)\|=\|y(i)\|$. We infer from Equation (1) that

$$
\begin{aligned}
&\{\|2x\|+\|y\|,\|y\|\}\\
&=\{\|V\{x(i)+y(i)\}+V\{x(i)\}\|,\|V\{x(i)+y(i)\}+V\{y(i)\}\|\}\\
&=\left\{\sum_{i=1}^{n}\frac{\|x'(i)\|}{\lambda_i}V\left(\lambda_i e_{\frac{x'(i)}{\|x'(i)\|}}\right)+\sum_{i=1}^{n}\frac{\|y'(i)\|}{\mu_i}V\left(\mu_i e_{\frac{y'(i)}{\|y'(i)\|}}\right)+\sum_{i=1}^{n}\frac{\|x(i)\|}{\lambda_i}V\left(\lambda_i e_{\frac{x(i)}{\|x(i)\|}}\right),\right.\\
&\quad\left.\sum_{i=1}^{n}\frac{\|x'(i)\|}{\lambda_i}V\left(\lambda_i e_{\frac{x'(i)}{\|x'(i)\|}}\right)+\sum_{i=1}^{n}\frac{\|y'(i)\|}{\mu_i}V\left(\mu_i e_{\frac{y'(i)}{\|y'(i)\|}}\right)+\sum_{i=1}^{n}\frac{\|y(i)\|}{\mu_i}V\left(\mu_i e_{\frac{y(i)}{\|y(i)\|}}\right)\right\}\\
&=\left\{\sum_{i=1}^{n}\frac{\|x'(i)\|}{\lambda_i}V\left(\lambda_i e_{\frac{x'(i)}{\|x'(i)\|}}\right)+\sum_{i=1}^{n}\frac{\|x(i)\|}{\lambda_i}V\left(\lambda_i e_{\frac{x(i)}{\|x(i)\|}}\right)+\|\{y(i)\}\|,\right.\\
&\quad\left.\sum_{i=1}^{n}\frac{\|x'(i)\|}{\lambda_i}V\left(\lambda_i e_{\frac{x'(i)}{\|x'(i)\|}}\right)-\sum_{i=1}^{n}\frac{\|x(i)\|}{\lambda_i}V\left(\lambda_i e_{\frac{x(i)}{\|x(i)\|}}\right)+\|\{y(i)\}\|\right\}
\end{aligned}
$$

Through the above equation we can get $\|x'(i)\|=\|x(i)\|$ or $\|x'(i)\|=-\|x(i)\|$. The proof is complete.□

Lemma 3.5. Let $X=s_n(H)$ and $Y=s_n(H)$. Suppose that $V:X\to Y$ is a surjective mapping satisfying Equation (1). Then V is injective and $V(-x)=-V(x)$ for all $x\in X$.

Proof: Suppose that V is surjective and $V(x)=V(y)$ for some $x,y\in X$. Putting $y=x$ in the Equation (1), this yields

$$\{\|2V(x)\|,0\}=\{\|2x\|,0\}$$

$V(x)=0$ if and only if $x=0$. Assume that $V(x)=V(y)\neq 0$ choose $z\in X$ such that $V(z)=-V(x)$, using the Equation (1) for x,y,z, we obtain

$$\{\|x+y\|,\|x-y\|\}=\{\|V(x)+V(y)\|,\|V(x)-V(y)\|\}=\{\|2V(x)\|,0\}$$

$$\{\|x+z\|,\|x-z\|\}=\{\|V(x)+V(z)\|,\|V(x)-V(z)\|\}=\{\|2V(x)\|,0\}$$

This yields $y,z\in\{x,-x\}$. If $z=x$, then $V(x)=-V(x)=0$, which is a contradiction. So we obtain $z=-x$, and we must have $y=x$. For otherwise we get $y=z=-x$ and

$$V(x)=V(y)=V(z)=-V(x)=0$$

This lead to the contradiction that $V(x)\neq 0$.

Theorem 3.6. Let $X=s_n(H)$ and $Y=s_n(H)$. Suppose that $V:X\to Y$ is a surjective mapping satisfying Equation (1). Then V is phase equivalent to a linear isometry *J*.

Proof: Fix $\gamma_0\in\Gamma$, and let $Z=\left\{z\in X:z\perp e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right\}$. By Lemma 3.4 we can write

$$V\left(z+\lambda e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right)=\alpha(z,\lambda)V(z)+\beta(z,\lambda)V\left(\lambda e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right),|\alpha(z,\lambda)|=|\beta(z,\lambda)|=1$$

for any $z \in Z$. Then, we can define a mapping $J : s_n(H) \to s_n(H)$ as follows:

$$J\left(z + \lambda e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right) = \alpha(z,\lambda)\beta(z,\lambda)V(z) + V\left(\lambda e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right)$$

$$J(\lambda z) = \alpha(z,\lambda)\beta(z,\lambda)V(\lambda z)$$

$$J\left(e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right) = V\left(e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right), \quad J\left(-e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right) = -V\left(e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right)$$

for $\forall 0 \neq \lambda \in \mathbb{R}$. The J is phase equivalent to V. So it is easily to know that J satisfies functional Equation (1). For any $z \in Z$, and $\forall 0 \neq \lambda \in \mathbb{R}$,

$$\begin{aligned}
&\left\{\|2z\| + \frac{1}{2^{\gamma_0}}\frac{|1+\lambda|}{1+|1+\lambda|}, \frac{1}{2^{\gamma_0}}\frac{|1-\lambda|}{1+|1-\lambda|}\right\} \\
&= \left\{\left\|J\left(z + e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right) + J\left(z + \lambda e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right)\right\|, \left\|J\left(z + e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right) - J\left(z + \lambda e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right)\right\|\right\} \\
&= \left\{\left\|\alpha(z,1)\beta(z,1)V(z) + \alpha(z,\lambda)\beta(z,\lambda)V(z) + V\left(e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right) + V\left(\lambda e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right)\right\|,\right. \\
&\quad \left.\left\|\alpha(z,1)\beta(z,1)V(z) - \alpha(z,\lambda)\beta(z,\lambda)V(z) + V\left(e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right) - V\left(\lambda e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right)\right\|\right\} \\
&= \left\{\left\||\alpha(z,1)\beta(z,1) + \alpha(z,\lambda)\beta(z,\lambda)|V(z)\right\| + \frac{1}{2^{\gamma_0}}\frac{|1+\lambda|}{1+|1+\lambda|},\right. \\
&\quad \left.\left\||\alpha(z,1)\beta(z,1) - \alpha(z,\lambda)\beta(z,\lambda)|V(z)\right\| + \frac{1}{2^{\gamma_0}}\frac{|1+\lambda|}{1+|1+\lambda|}\right\}
\end{aligned}$$

That means $\alpha(z,1)\beta(z,1) = \alpha(z,\lambda)\beta(z,\lambda)$,

$J\left(z + \lambda e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right) = J(z) + V\left(\lambda e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right)$ for any $z \in Z$, and $\forall 0 \neq \lambda \in \mathbb{R}$.

That yields

$$\begin{aligned}
&\left\{\|J(z) + J(-z)\|, \left\|J(z) - J(-z) + 2V\left(e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right)\right\|\right\} \\
&= \left\{\left\|J\left(z + e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right) + J\left(-z - e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right)\right\|, \left\|J\left(z + e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right) - J\left(-z - e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right)\right\|\right\} \\
&= \left\{0, \left\|2\left(z + e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right)\right\|\right\}
\end{aligned}$$

That means $J(-z) = -J(z)$. On the other hand,

$$\left\{\left\|z_1+z_2\right\|+\frac{2}{3}\frac{1}{2^{\gamma_0}},\left\|z_1-z_2\right\|\right\}$$
$$=\left\{\left\|J\left(z_1+e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right)+J\left(z_2+e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right)\right\|,\left\|J\left(z_1+e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right)-J\left(z_2+e_{\frac{x(\gamma_0)}{\|\gamma_0\|}}\right)\right\|\right\}$$
$$=\left\{\left\|J(z_1)+J(z_2)\right\|+\frac{2}{3}\frac{1}{2^{\gamma_0}},\left\|J(z_1)-J(z_2)\right\|\right\}$$

for $\forall z_1, z_2 \in Z$, It follows that $\|J(x)-J(y)\|=\|x-y\|$ for all $x, y \in X$, by assumed conditions, so J is a surjective isometry.□

4. Conclusion

Through the analysis of this article, we can get the conclusion that if a surjective mapping satisfying phase-isometry, then it can phase equivalent to a linear isometry in the space s and the space $s(H)$.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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