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Weierstrass's Global Division Theorem and Continuity of Linear Operators in H-spaces

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Abstract

In the present article generalization of Weierstrass's preparation theorem and the division theorem for germs of holomorphic functions at a point of n-dimensional complex space are considered. The author formulates the global theorem about division in terms of existence and continuity of the linear operator.

Keywords: *Weierstrass's global division theorem, closed graph theorem, holomorphic functions, H-space.*

1 Introduction

Generalization of the concepts of direct and inverse spectra of objects of an additive semiabelian category ***G*** (in the sense V.P.Palamodov) was introduced by E.Smirnov [1]: the concept of a Hausdorff spectrum, analogous to the δ_s-operation in descriptive set theory, so was solved Grothendieck's problem about iterated classes of locally convex spaces for closed graph theorem . This idea is characteristic even for algebraic topology, general algebra, global analysis, category theory and the theory of generalized functions. The construction of Hausdorff spectra $\boldsymbol{X}=\{X_s,\boldsymbol{F},h_{s's}\}$ is achieved by successive standard extension of a small category of indices Ω. The category ***H*** of Hausdorff spectra turns out to be additive and semiabelian under a suitable definition of spectral mapping. In particular, ***H*** contains V. P. Palamodov's category of countable inverse spectra with values in the category *TLG* of locally convex spaces. The *H*-limit of a Hausdorff spectrum in the category *TLG* generalizes the concepts of projective and inductive limits and is defined by the action of the functor Haus: $\boldsymbol{H} \to TLC$. The class of *H*-spaces is defined by the action of the functor Haus on the countable Hausdorff spectra over the category of Banach spaces; the closed graph theorem holds for its objects and it contains the category of Fréchet spaces and the categories of spaces due to De Wilde, D. A. Rajkov and E. Smirnov . The *H*-limit of a Hausdorff spectrum of *H*-spaces is an *H*-space.

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Weierstrass's preparation theorem and the division theorem for germs of holomorphic functions at a point $w \in \mathbb{C}^n$ allow us to establish a series of properties of local rings ${}_nO_w$ and modules over these rings (Noetherian, Oka's Lemma on the exactness of homomorphisms of ${}_nO$-modules, etc. [2]). The proofs have a number of algebraic characteristics, therefore consideration of a global variant of the theorems is significantly different and uses topological results of linear analysis . A more careful analysis makes it possible to formulate a global division theorem in terms of the existence and continuity of a linear operator acting on locally convex spaces so that the local and global variants of Weierstrass's theorem turn out to be in fact special cases of a more general theorem. In this article we obtain a stronger form of Theorems II.B.3 and II.D.1 in [2] for the case of H-spaces. $\mathbb{C}_m^{n-1}$ denotes $\underbrace{\mathbb{C}\times\ldots\times\mathbb{C}}_{m-1}\times\underbrace{\mathbb{C}\times\ldots\times\mathbb{C}}_{n-m}$ and $\pi_m : \mathbb{C}^n \to \mathbb{C}_m^{n-1}$ is the projection of $\mathbb{C}^n$ onto $\mathbb{C}_m^{n-1}$; at the same time $\pi^m : \mathbb{C}^n \to \mathbb{C}_m$ and $\mathbb{C}^n = \mathbb{C}_m^{n-1} \times \mathbb{C}_m$ so that π^m is the projection of $\mathbb{C}^n$ onto $\mathbb{C}_m$. For notational convenience in what follows the germ of a holomorphic function is denoted by capital Roman letters F , G , H, ...

2 Methodology and Methods

Let $\{S_U, \rho_{UV}\}$ be a presheaf of abelian groups over a topological space D, Ω a nonempty partially ordered set and F an admissible class for Ω (we may assume without loss of generality that $\Omega = |\mathrm{F}|$). Let us denote by $\hat{H}(S)$ a covariant functor from $\operatorname{Ord}\Omega$ to $\operatorname{Ord}U$, where U is a base of open sets in D, and by $\check{H}(S)$ a contravariant functor from $\operatorname{Ord}U$ to the category of abelian groups so that an abelian group S_U is defined for each $U \in U$ and a homomorphism $\rho_{UV} : S_U \to S_V$ is defined for each pair $U \subset V$. Then $H = \check{H}(S) \circ \hat{H}(S)$ is a contravariant functor of the Hausdorff spectrum $X(S) = \{S_{U_s}, \mathrm{F}, \rho_{U_s U_s}\}$, which we will call the *Hausdorff spectrum associated with the presheaf* $\{S_U, \rho_{UV}\}$. Let X be the H-limit of the Hausdorff spectrum $X(S)$ in the category of abelian groups and let

$$A = \bigcap_{F \in \mathrm{F}} \bigcup_{s \in |F|} U_s .$$

The following statement is proved in [3].

Proposition 1. *Let S be the sheaf of germs of holomorphic functions on an open set $D \subset \mathbb{C}^n$, associated with the presheaf $\{S_U, \rho_{UV}\}$, and let $X(S) = \{S_{U_s}, \mathrm{F}, \rho_{U_s U_s}\}$ be the associated true Hausdorff spectrum. Then the H-limit of the Hausdorff spectrum $X(S)$ is isomorphic to the vector space of sections $\Gamma(A, S)$ of the sheaf S over the set A.*

Proposition 2. *Let* $X(S)=\{\Gamma(U_s,S),\mathrm{F},\rho_{U_s U_s}\}$ *be a true countable Hausdorff spectrum and suppose that* $A=\bigcap_{\mathrm{F}}\bigcup_{F}U_s$ *has a countable fundamental system of compact sets, is connected and* $\mathring{A}\neq\varnothing$. *Then the* H*-limit* $X=\underset{\overrightarrow{\mathrm{F}}}{\overleftarrow{\lim}}\,\rho_{U_s U_s}\Gamma(U_s,S)$ *is a separated* H*-space in the topology* τ^* *and is continuously embedded in* O_A $(O_A$ *is the algebra of holomorphic functions on* $A)$.

Proof. First of all, by Proposition 1 we have the isomorphism $H:X\to\Gamma(A,S)$; because of the connectedness of A and the fact that $\mathring{A}\neq\varnothing$ each holomorphic function on A, $\varphi\in O_A$, is generated by some holomorphic function on the open set $U(\varphi)$; moreover, any two holomorphic functions $\varphi_1\in O_U$ and $\varphi_2\in O_V$ $(U\supset A, V\supset A)$ which coincide on A must coincide on a connected component of the intersection $U\cap V$ (see [1, p. 104]), which also implies the isomorphism $\Gamma(A,S)\equiv O_A$. Since A has a countable fundamental system of compact subsets

$$K_n \quad (n=1,2,\ldots), \quad K_1\subset K_2\subset\ldots,$$

on putting

$$\|\varphi\|_n=\max_{z\in K_n}|\varphi(z)| \quad (\varphi\in O_A),$$

we obtain a seminorm on O_A (or on $\Gamma(A,S)$, which is permissible according to the construction). Furthermore, on putting

$$p(\varphi)=\sum_{n=1}^{\infty}2^{-n}\frac{\|\varphi\|_n}{1+\|\varphi\|_n} \quad (\varphi\in O_A)$$

for example, we obtain a quasinorm on O_A under which O_A becomes a separated locally convex space with a countable base of neighbourhoods of zero, therefore metrizable, but in general not complete; we will denote this space by (O_A,p).

We now show that on O_A the locally convex topology τ^* of the H-limit of the Hausdorff spectrum $X(S)$ is not weaker than p . In fact, let $W=\{\varphi\in O_A:\|\varphi\|_N<\varepsilon\}$ be some neighbourhood of zero in (O_A,p) and let $F\in\mathrm{F}$. Let us choose $s_0\in|F|$ such that

$U_{s_0} \supset K_N$ – this choice turns out to be possible because of the compactness of K_N and the condition $A \subset \bigcup_F U_s$; also we can find a compact set $K_m^0 \subset U_{s_0}$ such that $K_m^0 \supset K_N$ – here the choice is possible because of the availability of a fundamental system $\{K_n^0\}_{n=1}^{\infty}$ in U_{s_0}.

$$H \circ \psi(M^F) \subset W,$$

where

$$\xi = (f_s)_F, M^F = \{\xi \in V_F^{s_0} : \sup_{z \in K_m^0} | f_{s_0}(z) | < \varepsilon\}, H \circ \psi(\xi) = \varphi, f_{s_0}|_A = \varphi.$$

Since $\psi(M^F)$ is itself a neighbourhood of zero in the MVG $X_{(F)}$ and $F \in \mathrm{F}$ was chosen arbitrarily, we have that

$$H(\mathrm{co}\bigcup_{\mathrm{F}} \psi(M^F)) \subset W$$

and is a neighbourhood of zero in the topology τ^*. This also shows that $\tau^* \geq p$. The proposition is proved.

Thus we obtain the following

Proposition 3. *Every connected bounded subset* $A \subset \mathrm{C}^n$ *has a representation*

$$A = \bigcap_{F \in \mathrm{F}} \bigcup_{s \in F} U_s, \tag{1}$$

where F *is an admissible class for the countable set* Ω *and the* U_s *are connected open subsets* (*domains*) *in* C^n.

In particular, for such a set A the Hausdorff spectrum

$$X(S) = \{\Gamma(U_s, S), \mathrm{F}, \rho_{U_{s'} U_s}\}$$

is true (it suffices to apply the uniqueness theorem for holomorphic functions). In the representation (1) it is natural to require that if $U_s \cap U_{s'} \neq \varnothing$ $(s, s' \in |\mathrm{F}|)$ then it is a connected set. Only such sets A will be considered further.

In what follows the space O_A of germs of holomorphic functions on A will be provided with the topology p (in general not separated) of uniform convergence on the compact subsets of A and with the locally convex topology of the H -limit. As has already been noted above (Proposition 1), for a connected bounded subset $A \subset \mathbb{C}^n$ we have the linear isomorphism

$$X \equiv \Gamma(A,S) \equiv O_A .$$

We also note that if the set A has an interior point then O_A coincides with the space of holomorphic functions on A (up to isomorphism).

2.1 Weierstrass's Global Division Theorem

We will say that the germ $H \in O_A$ $(A \subset \mathbb{C}^n)$ is a *w-local Weierstrass polynomial in z_m* $(1 \le m \le n)$ *of degree k* $(k > 0)$ if there exists $w \in A$ and a function $h \in H$ which is holomorphic on an open neighbourhood $U \supset A$ and has representation on U

$$\begin{aligned} h(z) &= (z_m - w_m)^k + a_1(z')(z_m - w_m)^{k-1} + \ldots + a_k(z'), \\ z' &= (z_1, z_2, \ldots, z_{m-1}, z_{m+1}, \ldots, z_n), \end{aligned} \tag{2}$$

where the $a_j(z')$ are holomorphic functions on $\pi_m(U)$, $a_j(w') = 0$, and $w = w' \times w_m$ $(j = 1, 2, \ldots, k)$. It is clear that the holomorphic function h is regular of order k in z_m at the point $w \in A$.

Theorem 1. (Weierstrass's global division theorem.) *Let $A \subset \mathbb{C}^n$ be a nonempty connected bounded set such that $\pi^m(A)$ is closed and let $H \in O_A$ be a w-local Weierstrass polynomial in z_m of degree k $(k > 0)$ with representation $h_U \in H$ such that*

$$\{z \in \pi_m^{-1} \circ \pi_m(A) \cap U : h_U(z) = 0\} \subset A.$$

Then there exists a continuous linear operator $L : O_A \to O_A \times O_A$, where

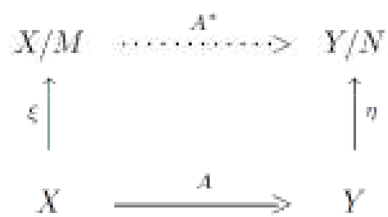
$$L(F) = (G, P), \qquad F = GH + P,$$

$$P = \sum_{j=0}^{k-1} P_j(z') z_m^j, \qquad P_j \in O_A .$$

First of all we recall [2, Chapter 2, §5] that O_A has the topology p of uniform convergence on the compact subsets, which in general is neither separated nor complete, and $O_A \times O_A$ has the usual product topology. In the course of the proof of Theorem 1 O_A will also be given another stronger locally convex topology, again in general not separated, under which it is an H-space. Therefore we first present a lemma for Theorem 1.

Lemma 1. *Let $A: X \to Y$ be a closed linear operator, where X is an H-space under the locally convex topology τ^* and (Y,σ) is an H-space $($in general X, Y are not separated spaces$)$. Then A is continuous.*

Proof. Let M, N be the respective nonseparated parts of X, Y and X/M, Y/N the separated quotient spaces with quotient maps $\xi: X \to X/M$ and $\eta: Y \to Y/N$. Then the quotient topology $\xi\tau^*$ on X/M is in general weaker than the topology $(\xi\tau^*)^*$, the limit of the corresponding Hausdorff spectrum (see, for example, [4]); let $\eta\sigma$ be the quotient topology on Y/N. Then the diagram



(3)

is commutative and the induced mapping A * exists because of the closedness of the operator A. In fact, the closedness of A implies that $N = \bigcap_{U\in U, V\in \mathbf{V}} \{U + AV\}$, where U, AV are bases of neighbourhoods of zero for the topologies σ, $A\tau^*$ respectively. But $AM \subset AV$ for any $V \in$ **V**. And $0 \in U$, therefore $AM \subset U + AV$ $(\forall U, V)$ and, consequently, $AM \subset N$. Moreover, the induced mapping A^* is clearly linear; we will show that A^* is a closed operator. For this we have to show that

$$0 = \bigcap_{U\in U, V\in \mathbf{V}} \{\eta U + A^*\xi V\}.$$

Since $\eta A = A^*\xi$, this is equivalent to the relation $0 = \bigcap_{U,V} \eta\{U + AV\}$. Let us suppose that $a \in \bigcap_{U,V} \eta\{U + AV\}$, then $\eta^{-1}a \cap (U + AV) \neq \varnothing$ $(\forall U,V)$. But using the closeness of *A* and for some $y \in U$ we have $\eta^{-1}a = y + N$ and because of the absolute convexity of $U + AV$ and Theorem 1.3 of [3] we obtain $\eta^{-1}a \subset U + AV$ $(\forall U,V)$. This implies that $\eta^{-1}a \subset N$; consequently $a = 0$ and A^* is a closed operator.

Thus by the Closed Graph Theorem for the H-space $(Y/N, \eta\sigma)$ and complete MVGs the closed operator A^* is continuous from $(X/M, (\xi\tau^*)^*)$ to $(Y/N, \eta\sigma)$. The existence of the Hausdorff spectrum for $(Y/N, \eta\sigma)$ follows from Proposition 4.10 and [4].

Now we will establish the continuity of the operator $A: X \to Y$. Let W be a closed absolutely convex neighbourhood of zero in Y and (V_n^F) a base of absolutely convex neighbourhoods of zero in the TVG $X_{(F)}$ $(F \in \mathrm{F})$, where

$$X = \bigcup_{F \in \mathrm{F}} \bigcap_{s \in F} X_s .$$

If it is shown that $A: X_{(F)} \to Y$ is continuous, then by the definition of the topology τ^* and the local convexity of (Y, σ) this will imply that $A: X \to Y$ is continuous. Therefore let $F \in \mathrm{F}$ be fixed. Then (ξV_n^F) is a base of neighbourhoods of zero for the TVG $(X/M)_{(F)}$ (see Proposition 4.10) and ηW is a neighbourhood of zero in $(Y/N, \eta\sigma)$. By the commutativity of Diagram (3) $A^*\xi V_n^F = \eta A V_n^F$ $(\forall n \in \mathrm{N})$ and by the continuity of A^* there exists $\overline{N} \in \mathrm{N}$ such that $A^*\xi V_{\overline{N}}^F \subset \eta W$ or $\eta A V_{\overline{N}}^F \subset \eta W$. Hence, $A V_{\overline{N}}^F \subset W + N$, but since W is a closed set and $N \subset W$, then $W + N \subset W$ and the continuity of $A: X_{(F)} \to Y$ is established. This means that $A: X \to Y$ is continuous and the lemma is proved.

Lemma 2. *Let* $L:(O_A, p) \to (O_A, p)$ *be a closed linear operator. Then* $L:(O_A, p^*) \to (O_A, p^*)$ *is continuous* (A *is a nonempty connected bounded subset of* C^n).

Proof. We recall that the locally convex topology p^* of the H-limit of a Hausdorff spectrum on the space of germs of holomorphic functions on A is not weaker than the locally convex topology p of uniform convergence on the compact subsets of A. Therefore the operator $L:(O_A, p^*) \to (O_A, p^*)$ is closed. Moreover, by Proposition 3.10 the set A has a representation

$$A = \bigcap_{F \in \mathrm{F}} \bigcup_{s \in F} U_s ,$$

where F is an admissible class for the countable set Ω and U_s is a domain in C^n; moreover, each U_s $(s \in |\mathrm{F}|)$ has a countable fundamental system of compact subsets $(K_n^s)_{n=1}^{\infty}$ with $K_1^s \subset K_2^s \subset \dots$. We will show that each space $(O_A)_{(F)}$ $(F \in \mathrm{F})$ is complete and so (O_A, p^*) is an H-space.

We recall that

$$X = \bigcup_{F \in \mathrm{F}} \bigcap_{s \in F} \psi(V_F^s)$$

and $\kappa : X \to \Gamma(A, S) \equiv O_A$ is an isomorphism. The TVG $(O_A)_{(F)}$ is an isomorphic image of the restriction of the complete TVG of countable character $S_{(F)}$ (notation of 3.2) to X. Therefore it is enough to establish the closedness of $\kappa^{-1}(O_A)_{(F)}$ in $S_{(F)}$. The arguments are carried out more easily for the germs of holomorphic functions on A.

Let $F \in \mathrm{F}$, $U_F \supset A$, $U_F = \bigcup_{s \in F} U_s$ (F is no more than countable and is totally linearly ordered for s). Further, let (G_n) be a sequence of germs of holomorphic functions on A which is fundamental in $(O_A)_{(F)}$. Since $(O_A)_{(F)}$ is a quotient group (up to isomorphism) of the complete MVG $(\prod_F O_{U_s})_{(F)}$, where O_{U_s} is the Fréchet space with the topology of uniform convergence on the compact sets $(K_n^s)_1^\infty$, it follows from Proposition 4.10 that there exists a subsequence $g_{n_k} \in \prod_F O_{U_s}$ $(k = 1, 2, \ldots)$ such that g_{n_k} converges in $(\prod_F O_{U_s})_{(F)}$ to some element $g \in \prod_F O_{U_s}$ and $\psi g_{n_k} = G_{n_k}$ $(k = 1, 2, \ldots)$. The last condition implies in particular that $g_{n_k} = (f_s^{n_k})_{s \in F}$, where $f_p^{n_k}|_{U_s} = f_s^{n_k}$ $(s \le p, p \in F)$, $p = p(k)$, $(k = 1, 2, \ldots)$. Put $p_0 = \inf_k p(k)$, $p_0 \in F$. Then, clearly, $f_{p_0}^{n_k}|_{U_s} = f_s^{n_k}$ $(s \le p_0, k = 1, 2, \ldots)$; we will denote by $f_k = f_{p_0}^{n_k}$ the holomorphic functions on the open connected set U_{p_0} $(k = 1, 2, \ldots)$. Since $\lim_{k \to \infty} g_{n_k} = g$ and $g = (\hat{g}_s)_{s \in F}$, then, in particular, f_k converges to $\hat{g}_{p_0}$ in $O_{U_{p0}}$ and moreover $g - g_{n_{k_l}} \in V_F^s$ $(s \in F, n_{k_l} = n_{k_l}(s), l = 1, 2, \ldots)$. The last observation means that for $s > p_0$ the holomorphic function $\hat{g}_{p_0} - f_{n_{k_l}}$ has a unique extension to the set U_s $(s \in F)$. However, each element g_{n_k} is equivalent to elements $a_k \in \prod_{F_k} O_{U_s}$, i.e. $\psi g_{n_k} = \psi a_k$ and moreover $a_k \in \bigcap_{s \in F_k} V_{F_k}^s$ $(k = 1, 2, \ldots)$. Furthermore, we may assume without loss of generality that $F_1 \prec F_2 \prec \ldots$. Thus the holomorphic function $f_{n_{k_l}}$ has a unique extension to the set $U_{F_{k_l}} \supset A$ $(l = 1, 2, \ldots)$ and, consequently, the holomorphic function $\hat{g}_{p_0}$ has a unique extension to the set $U_s \cap U_{F_{k_l}}$ $(s > p_0, s \in F)$. Since $U_s \cap U_{F_{k_l}} = \bigcup_{q \in F_{k_l}} (U_s \cap U_q)$ and $U_s \cap U_q \subset U_s \cap U_{q'}$ $(q \le q')$, then $U_s \cap U_{F_{k_l}}$ is a connected open set

$(l = 1, 2, \ldots, k_l = k_l(s))$, and since the set $\{s \in F : s > p_0\}$ can be enumerated, let its points be $s_1, s_2, \ldots$.

Thus on each nonempty open connected set $U_s \cap U_{F_{k_l}}$ a holomorphic function $\hat{g}_{p_0 s}$ is defined such that $\hat{g}_{p_0 s}|_{U_{p_0}} = \hat{g}_{p_0} \quad (s > p_0, s \in F)$. But since each nonempty intersection $(U_{s_i} \cap U_{F_{k_i}}) \cap (U_{s_j} \cap U_{F_{k_j}})$ is connected by construction and has nonempty intersection with U_{p_0}, then a holomorphic function $\hat{g}$ is defined on the open set $\bigcup_{i=1}^{\infty} (U_{s_i} \cap U_{F_{k_i}})$ such that $\hat{g}|_{U_{s_i} \cap U_{F_{k_i}}} = \hat{g}_{p_0 s_i} \quad (i = 1, 2, \ldots)$. Then $\hat{g}|_{U_{F^*}}$ generates an element of $\bigcap_{s \in F^*} V_{F^*}^s$ such that $\psi g = \psi \hat{g}|_{U_F}$ and, consequently, $\psi g = G \in O_A$ and $\lim_{n \to \infty} G_n = G$ in the TVG $(O_A)_{(F)}$. Thus the space $(O_A)_{(F)}$ is complete and (O_A, p^*) is an H-space ($U_{F^*} \subset \bigcup_{i=1}^{\infty} (U_{s_i} \cap U_{F_{k_i}}), F^* \in \mathrm{F}$).

Continuity of the operator A now follows from the Closed Graph Theorem, Lemma 1 and the closedness of the operator $A : (O_A, p^*) \to (O_A, p^*)$. The lemma is proved.

Proof of Theorem 1. Let $H \in O_A$ be a w-local Weierstrass polynomial in z_m of degree k and let $h \in H$ be a holomorphic function on the open connected set $U \supset A$ which satisfies the conditions of the theorem and the relation (2). Furthermore, let $F \in O_A$ be an arbitrary germ, let $f \in F$ and suppose that f is a holomorphic function on the domain $V \subset U_1$ (it may be assumed without loss of generality that $\overline{U}_1 \subset U$). Let us fix a point $a' \in \pi_m(A)$ and a closed (according to the condition) cross-section $r_{a'}(A) \subset r_{a'}(U)$ and choose a closed piecewise-smooth Jordan contour $\Gamma_{a'}$ which encloses $r_{a'}(A)$ and lies in $r_{a'}(V)$ and has length $l(\Gamma_{a'})$. Since the function $h(z)$ is continuous on the open neighbourhood of the compact set

$$Q = \{z \in V : z_m \in \Gamma_{a'}, \pi_m(z) = a'\},$$

there exists an open ball $B(0, \delta)$ such that for $z_m \in \Gamma_{a'}$ and $z' \in \pi_m[(a', z_m) + B(0, \delta)]$ we have the inequality

$$|h(z) - h(a', z_m)| \leq \inf_{r_{a'}} |h(a', z_m)| \tag{4}$$

and the inclusion

$$(a', z_m) + B(0,\delta) \subset V \quad (z_m \in \Gamma_{a'}).$$

Moreover, by the compactness of Q we can choose a polydisk $\Delta'(0,\delta_{a'}) \subset C^{n-1}$ such that

$$[a' + \Delta'(0,\delta_{a'})] \times \Gamma_{a'} \subset \bigcup_{z_m \in \Gamma_{a'}} [(a', z_m) + B(0,\delta)]. \tag{5}$$

In fact, we cover the compact set Q with the open balls $(a', z_m) + B(0,\delta)$ $(z_m \in \Gamma_{a'})$, in each of which we choose a polydisk $(a', z_m) + \Delta(0,\delta_{a'})$ $(z_m \in \Gamma_{a'}, \delta_{a'} = (\delta_{a'}, \delta_m))$ with these taken together also covering Q. Put $R_{a'} = a' + \Delta'(0,\delta_{a'})$. Then

$$\bigcup_{z_m \in \Gamma_{a'}} [(a', z_m) + \Delta(0,\delta_{a'})] = [\bigcup_{r_{a'}} (z_m + \Delta(0,\delta_m))] \times R_{a'} \supset \Gamma_{a'} \times R_{a'},$$

from which (5) follows.

The inclusion (5) allows us to conclude in particular that (4) and the inclusion $\Gamma_{a'} \subset r_{z'}(V)$ are valid for $z' \in R_{a'}$. Now for the indicated $z' \in R_{a'}$ the function $h_{z'} = h(z', z_m)$ as a holomorphic function of one variable z_m has exactly k zeros inside the contour $\Gamma_{a'}$ by Rouché's Theorem for the domain $r_{z'}(V) \cap r_{a'}(V)$; in particular, $h_{z'} \neq 0$ on $\Gamma_{a'}$ and outside this contour in the domain $r_{z'}(V)$ (and even $r_{z'}(U)$).

We will denote by $D_{a'}$ the domain bounded by $\Gamma_{a'}$ and put

$$D = \bigcup_{a' \in \pi_m(A)} (D_{a'} \times R_{a'}).$$

It is clear that D is an open connected set such that $A \subset D \subset V \subset U_1$.

Further, for each open set $D_{a'} \times R_{a'}$ $(a' \in \pi_m(A))$ we define a holomorphic function (see [1])

$$g_{a'}(z) = \frac{1}{2\pi i} \int_{\Gamma_{a'}} \frac{f(z',\zeta)}{h(z',\zeta)} \cdot \frac{d\zeta}{\zeta - z_m}$$

and a holomorphic function $p_{a'}(z) = f(z) - g_{a'}(z) h(z)$. Therefore

$$p_{a'}(z) = \frac{1}{2\pi i} \int_{\Gamma_{a'}} \frac{f(z',\zeta)}{h(z',\zeta)} \left[\frac{h(z',\zeta) - h(z', z_m)}{\zeta - z_m} \right] d\zeta \ ,$$

where

$$p_{a'}(z) = \sum_{j=0}^{k-1} p_{a'_j}(z')(z_m - w_m)^j,$$

$$p_{a'_j} = \frac{1}{2\pi i}\int_{\Gamma_{a'}} \frac{h_j^*(z',\zeta)}{h(z',\zeta)} f(z',\zeta)\,d\zeta \quad (j = 0,1,\dots,k-1),$$

and the holomorphic functions h_j^* $(j = 0,1,\dots,k-1)$ are defined from (2) by consideration of the expression

$$\frac{h(z',\zeta) - h(z',z_m)}{\zeta - z_m}.$$

The uniqueness of the functions $g_{a'}$ and $p_{a'}$ is established similarly to [1, p. 93] by using Rouché's Theorem.

If $(D_{a'} \times R_{a'}) \cap (D_{a''} \times R_{a''}) \neq \varnothing$, then for $\hat{z} \in (D_{a'} \times R_{a'}) \cap (D_{a''} \times R_{a''})$ we have $\hat{z}_m \in D_{a'} \cap D_{a''}$. Because the contours $\Gamma_{a'}$ and $\Gamma_{a''}$ are homotopic this implies that the following identity holds:

$$\int_{\Gamma_{a'}} \frac{f(\hat{z}',\zeta)}{h(\hat{z}',\zeta)} \cdot \frac{d\zeta}{\zeta - \hat{z}_m} = \int_{\Gamma_{a''}} \frac{f(\hat{z}',\zeta)}{h(\hat{z}',\zeta)} \cdot \frac{d\zeta}{\zeta - \hat{z}_m}.$$

Thus $g_{a'}(\hat{z}) = g_{a''}(\hat{z})$ and, consequently, a holomorphic function $g(z)$ can be defined on the domain D such that $g|_{R_{a'} \times D_{a'}} = g_{a'}$ $(a' \in \pi_m(A))$. In the same way a holomorphic function $p(z)$ can be defined such that $p|_{R_{a'} \times D_{a'}} = p_{a'}$ $(a' \in \pi_m(A))$ and

$$p(z) = \sum_{j=0}^{k-1} p_j(z')(z_m - w_m)^j,$$

so that we have the unique representation

$$f(z) = g(z)h(z) + p(z) \quad (z \in D). \tag{6}$$

Thus a linear operator $L : O_A \to O_A \times O_A$ is defined by the relation $L(F) = (G,P)$, $F = GH + P$, $f \in F$, $g \in G$, $h \in H$, $p \in P$. The operator L has components

$L_1 : F \to G$ and $L_2 : F \to P$, whose continuity in the respective topologies will also imply that of L. Let us therefore investigate the continuity of the operators L_1 and L_2.

It follows clearly from the relation (6) that L_1 and L_2 are closed linear operators from (O_A, p) into (O_A, p). Thus by Lemma 2 the operator $L_i : (O_A, p^*) \to (O_A, p^*)$ is continuous $(i = 1, 2)$, as also is the operator

$$L : (O_A, p^*) \to (O_A, p^*) \times (O_A, p^*).$$

We now establish the continuity of the operator $L : (O_A, p) \to (O_A, p) \times (O_A, p)$. First of all, let us fix an open set D^h constructed by the method indicated above for the holomorphic function $h(z)$ on the domain U_1 ; then by the compactness of A we choose a finite subcover $\bigcup_{i=1}^{N} (R_{a_i}^h \times D_{a_i}^h)$, where by the construction it may be assumed without loss of generality that on the distinguished boundary of the polydisk $R_{a_i}^h$ the function $h(z', \zeta) = 0$ only for $\zeta \in D_{a_i}^h$ $(i = 1, 2, \ldots, N)$. Therefore $h(z) \neq 0$ on the distinguished boundary of the polydomain $R_{a_i}^h \times D_{a_i}^h$, which is part of the boundary of the domain $\bigcup_{i=1}^{N} (R_{a_i}^h \times D_{a_i}^h) = U_2$. If we now put

$$M = \sup_{0 \le j \le k-1} \sup_{\bigcup_{i=1}^{N} (\overline{R_{a_i}^h} \times \Gamma_{a_i})} \left| \frac{h_j^*(z', \zeta)}{h(z', \zeta)} \right|,$$

then $M < +\infty$.

Now let $F \in O_A$, $L_1 : F \to G$, $L_2 : F \to P$ and choose $f \in F$ with domain of definition $V \subset U_2$; construct the domain $D^f \subset V$ such that $\overline{D^f} \subset V$, while the functions $p(z)$ and $g(z)$ are defined on D^f $(p \in P, g \in G)$ and the relation (6) holds. It is clear that $D^f \supset A$ and the following diagrams are commutative:

We will establish the continuity of the operator $L_2:(O_A, p)\to(O_A, p)$, the continuity of L_1 being obvious. Let $a'\in\pi_m(A)$. Then

$$\begin{aligned}|p_{a'}(z)| &\leq K\sum_{j=0}^{k-1}|p_{a_j}(z')|\leq\frac{K\cdot M}{2\pi}\sum_{j=0}^{k-1}\int_0^{l(\Gamma_{a'}^f)}|f(z',\zeta)|\cdot|d\zeta|\\ &\leq \frac{K\cdot M}{2\pi}\cdot k\cdot\sup_{\overline{D^f}}|f(z',\zeta)|\cdot l(\Gamma_{a'}^f).\end{aligned}$$

Now we choose a sequence $V_1=V\supset V_2\supset\ldots$ which is fundamental for A and compact sets $\overline{D}_m=\overline{D_m^f}$, where $\overline{D_m^f}\subset V_m$ such that $\overline{A}=\bigcap_{m=1}^{\infty}\overline{D_m^f}$ and, moreover, the sequence $(\overline{D}_m)$ converges to $\overline{A}$ in the Hausdorff metric for all compact subsets of $\mathbb{C}^n$. This means in particular that for $f\in O_{V_1}$ we have the relation

$$\overline{\lim_{m\to\infty}}\sup_{\overline{D}_m}|f(z)|\leq\sup_A|f(z)|.$$

In fact, let us assume the contrary, i.e. there exist $\varepsilon>0$ and a sequence (m_k) such that

$$\sup_A|f(z)|+\varepsilon<\sup_{\overline{D}_{m_k}}|f(z)|\quad(k\in\mathbb{N}).$$

From this we find a sequence (z_{m_k}) such that $z_{m_k}\in\overline{D}_{m_k}$ and

$$\sup_A|f(z)|+\varepsilon<|f(z_{m_k})|\quad(k\in\mathbb{N});$$

but then we can find a subsequence $(z_{m_{k_l}})$ such that $z^*=\lim_{l\to\infty}z_{m_{k_l}}$. Then $z^*\in\overline{A}$ and, consequently, we have the inequality

$$\sup_A|f(z)|+\varepsilon\leq|f(z^*)|,$$

which is impossible.

Therefore

$$\begin{aligned}\|P\|_A &= \sup_A |p(z)| \\ &\leq \overline{\lim_{m\to\infty}} \sup_{D_m} |p(z)| = \overline{\lim_{m\to\infty}} \sup_{D_m} |p_{a'}(z)| \\ &\leq \frac{K\cdot M\cdot k}{2\pi} \overline{\lim_{m\to\infty}} l(\Gamma_{a'}^m)\cdot \overline{\lim_{m\to\infty}} \sup_{\bar{D}_m} |f(z)| \\ &\leq K_A\cdot l_A\cdot \sup_A |f(z)| = K_A\cdot l_A\cdot \|F\|_A .\end{aligned}$$

Thus the operator $L_2 : (O_A, p) \to (O_A, p)$ is continuous. The theorem is proved.

3 Results and Discussion

We can use Weierstrass's global division theorem in the following situations:

1. Generalization of classical results of B. Malgrange [5] and L. Ehrenpreis [6] on the solvability of the unhomogeneous equation $p(D)D'=D'$, where $p(D)$ is a linear differential operator with constant coefficients in $\mathbf{R}^n$ and $D'=D'(S)$ is the space of generalized functions on a convex domain $S\subset \mathbf{R}^n$, can be extended to the case of sets S which are not necessarily open or closed.
2. Using homological methods one can establish the generalization of Palamodov's theorem [7] for vanishing at zero, $\text{Haus}^1(\boldsymbol{X})=0$, for the functor Haus of a Hausdorff limit associated with the representation (1), where $\boldsymbol{X}$ is the Hausdorff spectrum of the kernels of the operators $p(D){:}D'(T_s)\to D'(T_s)$ $(s\in|\boldsymbol{F}|)$. The condition $\text{Haus}^1(\boldsymbol{X})=0$ is equivalent to the condition that the operator $p(D){:}D'(S)\to D'(S)$ is an epimorphism.
3. We should learn a space of test functions on such sets $S\subset \mathbf{R}^n$ and prove that it is an H-space (generally nonmetrizable), that is

$$D(S) = \bigcup_{F\in \boldsymbol{F}} \bigcap_{s\in F} D(T_s), \tag{7}$$

   where $\{\cap_{s\in F}T_s\}_{F\in \boldsymbol{F}}$ forms a fundamental system of compact subsets of S and $D(T_s)$ is the Fréchet space of test functions with supports in the closed sets $T_s\subset\mathbf{R}^n$, where $S=\cup_{F\in\boldsymbol{F}}\cap_{s\in F}T_s$ (S – not necessarily open or closed).
4. We should prove the closed graph theorem for D(S) and $D'(S)$ for such S and consider some important applications to approximation theory [8-9].

4 Conclusion

Using of new concept of H-spaces in technique on the solvability of the unhomogeneous equation $p(D)D'=D'$, *where* $p(D)$ is a linear differential operator with constant coefficients in $\mathbf{R}^n$ and $D'=D'(S)$ is the space of generalized functions on a convex domain $S\subset \mathbf{R}^n$, we can extend it to the case of sets S which are not necessarily open or closed. So Weierstrass's global division theorem will be the main instrument for proving of an epimorphism of operator $p(D)$ and the search for a fundamental solutions of tempered distributions..

Competing interests

Authors have declared that no competing interests exist.

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