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A Modified-Form Expressions for the Hypoexponential Distribution

Khaled Smaili[1], Therrar Kadri[2] and Seifedine Kadry[3*]

[1]*Department of Applied Mathematics, Faculty of Sciences, Lebanese University, Zahle, Lebanon.*
[2]*Mathematics Department, Faculty of Science, Beirut Arab University, Beirut, Lebanon.*
[3]*School of Engineering, American University of the Middle East, Eguaila, Kuwait.*

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Abstract

The Hypoexponential distribution is the distribution of the sum of 2 independent Exponential random variables. This distribution is used in modeling multiple exponential stages in series. This distribution can be used in many domains of application. In this paper we find a modified and simple form of the probability density function for the general case of the Hypoexponential distribution when its parameters do not have to be distinct. This modified form is found by writing the probability density function of this distribution as a linear combination of the probability density function of the known Erlang distribution. Also, this modified form generates a simple form of the cummulative distribution function, moment generating function, reliability function, hazard function, and moment of order k for the general case of the Hypoexponential distribution. Moreover, new identities are established. Finally, we consider the coefficients of this linear combination and propose an algorithm to compute them.

Keywords: *Hypoexponential distribution; erlang distribution; probabilty density function; cummulative distribution function; moment generating function; reliability function; hazard function; expectation.*

1 Introduction

The sum of Random Variable (RV) plays an important role in modeling many events [1,2]. In particular the sum of exponential random variable has important applications in the modeling in many domains, such as communications and computer science [3,4], Markov process [5,6], insurance, [7,8] and reliability and performance evaluation [3,5,9]. Nadarajah [10], presented a review of some results on the sum of random variables.

Many processes in nature can be divided into sequential phases. If the time the process spends in each phase is independent and exponentially distributed, then the overall time is hypoexponentially distributed, see [3].

**Corresponding author: skadry@gmail.com;*

The Hypoexponential distribution is the distribution of the sum of $m \geq 2$ independent Exponential random variables. The general case of this distribution is when the m parameters do not have to be distinct. This general case can be written as$S_m = \sum_{i=1}^{n}\sum_{j=1}^{k_i} X_{ij}$, where X_{ij} is an Exponential RV of parameter α_i, written as $X_{ij} \rightsquigarrow Exp(\alpha_i)$, for $1 \leq i \leq n$ and $1 \leq j \leq k_i$ and $m = \sum_{i=1}^{n} k_i$. However, the case when $n = 1$ the m parameters are identical and the Hypoexponential distribution is the Erlang distribution [4]. Also the case when $k_i = 1$ for all $1 \leq i \leq n$, the m parameters are distinct, known to be the Hypoexponential distribution with different parameters discussed by Smaili et al. [11]. Moreover, the general case can also be considered as the sum of Erlang distributions. The study of this general case was discussed by many authors see [5,12].

In this paper, we find a modified and simple form of the PDF for the general case of the Hypoexponential distribution. This modified form is a linear combination of the PDF of the known Erlang distribution as $f_{S_m}(t) = \sum_{i=1}^{n}\sum_{j=1}^{k_i} A_{ij} f_{Y_{ij}}(t)$, where $f_{Y_{ij}}(t)$ is the PDF of the Erlang RV Y_{ij} of parameters j and α_i, written as $Y_{ij} \rightsquigarrow Erl(j, \alpha_i)$, and A_{ij} are the coefficients of this linear combination. Similarly, the CDF, MGF, reliability function and moment of order $k$for the general case of the Hypoexponential distribution is written as a linear combination of CDF, MGF, reliability function and moment of order k of the Erlang distribution respectively. So, in order to obtain a comprehensive study of S_m is from the known Erlang random variable Y_{ij}.

After introducing some definitions and notations in section 2, we expose the linear forms in Section 3. Moreover new formulas, equations and identities are presented. Finally in section 4, we design an algorithm for finding the coefficients of the linear combination, A_{ij}, stated in section 3. Moreover, we present the reliability and hazard functions in an example.

2 Definitions and Notations

Let X_{ij} be m independent Exponential RV, where the parameters α_i do not have to be distinct for $1 \leq i \leq n$ and $1 \leq j \leq k_i$ and $m = \sum_{i=1}^{n} k_i$. We define the random variable $S_m = \sum_{i=1}^{n}\sum_{j=1}^{k_i} X_{ij}$ to be the Hypoexponential random variable with parameters α_i and $k_i, i = 1,2,\ldots,n$, written as $S_m \rightsquigarrow Hypo\exp(\vec{\alpha}, \vec{k})$, where $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\vec{k} = (k_1, k_2, \ldots, k_n)$, which means that α_i has repeated k_i times. Also S_m can be considered to be the sum of n independent Erlang RV Z_{k_i} for $1 \leq i \leq n$ as

$$S_m = \sum_{i=1}^{n} Z_{k_i} \tag{2.1}$$

where $Z_{k_i} = \sum_{j=1}^{k_i} X_{ij}$ for $1 \leq i \leq n$.

Next, we state some notations and definitions used throughout the paper.

a. $X_{ij} \rightsquigarrow Exp(\alpha_i), Z_{k_i} \rightsquigarrow Erl(k_i, \alpha_i), Y_{ij} \rightsquigarrow Erl(j, \alpha_i)$.

b. f_X: the probability density function (PDF),F_X: cummulative distribution function (CDF), Φ_X: the moment generating function (MGF),R_X: the reliability function and h_X: the hazard function of the RV X.

c. $E[X^k]$ is the moment of order k of RV X.

d. $g_i^{(q)}(s)$: is the q^{th}derivative of the function $g_i(s) = \prod_{j=1,j\neq i}^{n} \frac{1}{(s+\alpha_j)^{k_j}}$.

3 Modified Form of the PDF and Related Functions for the General Case of the Hypoexponential Distribution

This section is divided into 4 subsections. In section 3.1, we find a modified and simple form of the PDF for the general case of the Hypoexponential distribution. In the second section, we deduce a new expression of the CDF and some equalities are obtained. The MGF and moment of order k were considered in Section 3.3. Some identities and formulas are obtained. The formulas are applied in two particular cases for $k = 1$ and $k = 2$. In the last section a modified form of the reliability and hazard functions are given.

3.1 PDF for the General Case of the Hypoexponential RV

In this part, we find a modified and simple form of the PDF as a linear combination of the PDF of the known Erlang distribution stated in Corollary 1. This modified form is applied on the two particular cases of the Hypoexponential random variable.

We start by stating in the following theorem, the PDF for the general case of the Hypoexponential distribution given by Jasiulewicz and Kordecki [5].

Theorem 1. *Let* $m \geq 2$ *and* $S_m \rightsquigarrow Hypoexp(\vec{\alpha},\vec{k})$ *where* $\vec{\alpha} = (\alpha_1,\alpha_2,\dots,\alpha_n)$ *and* $\vec{k} = (k_1,k_2,\dots,k_n)$. *Then,* $f_{S_m}(t) = \left(\prod_{i=1}^{n} \alpha_i^{k_i}\right) \sum_{i=1}^{n} \sum_{j=1}^{k_i} c_{ij} \frac{t^{j-1}e^{-\alpha_i t}}{(j-1)!} I_{(0,\infty)}(t)$,where

$$c_{ij} = \frac{1}{(k_i-j)!} \lim_{s\to-\alpha_i} g_i^{(k_i-j)}(s). \tag{3.1}$$

In the following Corollary we give the modified form of PDF for$Hypoexp(\vec{\alpha},\vec{k})$.

Corollary 1. *Let* $m \geq 2$ *and* $S_m \rightsquigarrow Hypoexp(\vec{\alpha},\vec{k})$ *where* $\vec{\alpha} = (\alpha_1,\alpha_2,\dots,\alpha_n)$ *and* $\vec{k} = (k_1,k_2,\dots,k_n)$. *Then*

$$f_{S_m}(t) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} A_{ij} f_{Y_{ij}}(t), \tag{3.2}$$

Where

$$A_{ij} = \left(\prod_{i=1}^{n} \alpha_i^{k_i}\right) \frac{c_{ij}}{\alpha_i^j} \tag{3.3}$$

and c_{ij} is defined in Eq (3.1).

Proof. From Theorem 1, the PDF of S_m can be written as

$$f_{S_m}(t) = \left(\prod_{i=1}^{n} \alpha_i^{k_i}\right) \sum_{i=1}^{n} \sum_{j=1}^{k_i} \frac{c_{ij}}{\alpha_i^j} \frac{(\alpha_i t)^{j-1}\alpha_i e^{-\alpha_i t}}{(j-1)!} I_{(0,\infty)}(t), \tag{3.4}$$

where c_{ij} given in Eq (3.1).

Now, we define that $A_{ij} = \left(\prod_{i=1}^{n} \alpha_i^{k_i}\right) \frac{c_{ij}}{\alpha_i^j}$ and $f_{Y_{ij}}(t) = \frac{(\alpha_i t)^{j-1} \alpha_i e^{-\alpha_i t}}{(j-1)!} I_{(0,\infty)}(t)$, for $1 \le i \le n$ and $1 \le j \le k_i$. However $f_{Y_{ij}}(t)$ is the PDF of the Erlang RV so called Y_{ij} having the parameters j and α_i. Therefore, we can rewrite Eq (3.4) in the following form $f_{S_m}(t) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} A_{ij} f_{Y_{ij}}(t)$.

Next, we present the two particular cases of $Hypoexp(\vec{\alpha}, \vec{k})$.

Case 1. $n = 1$, *The Erlang Distribution.*

In this case, we have the m exponential stages having identical parameters, which is the Erlang distribution. This case was presented by many authors as [12]. Thus for $n = 1, \vec{\alpha} = \alpha_1$ and $\vec{k} = k_1$, we may write $Hypoexp(\vec{\alpha}, \vec{k}) = Hypo\text{exp}(\alpha_1, k_1) = Erl(k_1, \alpha_1)$. Moreover, Corollary1 can be written in the form $f_{S_m}(t) = f_{Y_{1k_1}}(t)$. This form gives the following corollary.

Corollary 2. *Let* $S_m \sim Erl(k_1, \alpha_1)$. *Then* $A_{1j} = 0$ *for* $1 \le j \le k_1 - 1$ *and* $A_{1k_1} = 1$.

Proof. This case is when $n = 1$, then $m = k_1$ and from Eq (3.2) we obtain that $f_{S_m}(t) = \sum_{j=1}^{k_1} A_{1j} f_{Y_{1j}}(t)$. By comparing with $f_{S_m}(t) = f_{Y_{1k_1}}(t)$, we obtain that $A_{1j} = 0$ for $1 \le j \le k_1 - 1$ and $A_{1k_1} = 1$.

Case 2. $k_i = 1, i = 1,2,\dots,n$. *The Hypoexponential distribution of distinct parameters.*

This case was treated by Smaili et al. [11]. Having $k_i = 1, i = 1,2,\dots,n$, we may write $Hypo\text{exp}(\vec{\alpha}, \vec{1}) = Hypo\text{exp}(\vec{\alpha})$ and the PDF in Corollary1 is given in the following corollary.

Corollary 3. *Let* $S_m \sim Hypoexp(\vec{\alpha})$. *Then* $f_{S_m}(t) = \sum_{i=1}^{n} A_{i1} f_{Y_{i1}}(t)$ *where* $A_{i1} = \prod_{j=1, j\ne i}^{n} (\frac{\alpha_j}{\alpha_j - \alpha_i})$.

Proof. This case is when $k_i = 1$, thus $m = n$. From Eq (3.2), $f_{S_m}(t) = \sum_{i=1}^{n} A_{i1} f_{Y_{i1}}(t)$ and $Y_{i1} \sim Erl(1, \alpha_i) = Exp(\alpha_i)$. However, in [11], Smaili et al. presented the PDF of this case as $f_{S_m}(t) = \sum_{i=1}^{n} \prod_{j=1, j\ne i}^{n} \left(\frac{\alpha_j}{\alpha_j - \alpha_i}\right) f_{Y_{i1}}(t)$. Therefore, $A_{i1} = \prod_{j=1, j\ne i}^{n} (\frac{\alpha_j}{\alpha_j - \alpha_i})$.\

3.2 CDF for the Hypoexponential RV

In this part, we introduce a modified form of the CDF for the general case of the Hypoexponential RV. Moreover, an equality for the coefficients A_{ij} of the linear combination is obtained which shall be used later.

Corollary 4. *Let* $m \ge 2$ *and* $Y_{ij} \sim Erl(j, \alpha_i)$ *for* $1 \le i \le n$ *and* $1 \le j \le k_i$. *Then*

$$F_{S_m}(x) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} A_{ij} F_{Y_{ij}}(x) \quad (3.5)$$

Proof. The proof is a direct consequence of Corollary 1, knowing that CDF is the integral of the PDF. Hypoexponential RV.

In the next, proposition we determine an identity for the coefficients of the linear combination stated in Eq (3.2) using the above Proposition.

Proposition 1. $\sum_{i=1}^{n} \sum_{j=1}^{k_i} A_{ij} = 1$.

Proof. We have the cumulative of any random variable at infinity is 1. Then verifying this in Eq (3.5) gives the above result.

In the next Theorem, we use Propositions 1 and Corollary 4 to find a new form of CDF for the general case of Hypoexponential RV.

Theorem 2. *Let* $m \geq 2$. *Then* $F_{S_m}(x) = 1 - \sum_{i=1}^{n} \sum_{j=1}^{k_i} A_{ij} \frac{\Gamma(j,\alpha_i x)}{(j-1)!} I_{(0,\infty)}(x)$, *where* $\Gamma(a,z)$ *is the upper incomplete Gamma function [14].*

Proof. We have from Corollary 4, $F_{S_m}(x) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} A_{ij} F_{Y_{ij}}(x)$. Moreover, its known that the CDF of the Erlang distribution see [13], is given by $F_{Y_{ij}}(x) = 1 - e^{-\alpha_i x} \sum_{k=0}^{j-1} \frac{(\alpha_i x)^k}{k!} I_{(0,\infty)}(x)$, where $Y_{ij} \sim Erl(j, \alpha_i)$. Thus $F_{S_m}(x) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} A_{ij} \left(1 - e^{-\alpha_i x} \sum_{k=0}^{j-1} \frac{(\alpha_i x)^k}{k!}\right) I_{(0,\infty)}(x)$. But from Proposition 1, $\sum_{i=1}^{n} \sum_{j=1}^{k_i} A_{ij} = 1$ and $\sum_{k=0}^{j-1} \frac{(\alpha_i x)^k}{k!} = \frac{e^{\alpha_i x}\Gamma(j,\alpha_i x)}{(j-1)!}$, where $\Gamma(j, \alpha_i x)$ is the incomplete Gamma function see [14]. Therefore, we obtain that result.

3.3 MGF and Moment of Order k for the Hypoexponential RV

In this part, we introduce a modified form of the MGF for the general case of the Hypoexponential RV. Next, we introduce two new forms of the Moment of S_m of order k of the general case of the Hypoexponential RV. These new forms are compared to determine a generalized equality. Our results are applied on the two particular cases $k = 1$ and $k = 2$.

Corollary 5. *Let* $m \geq 2$ *and* $Y_{ij} \sim Erl(j, \alpha_i)$ *for* $1 \leq i \leq n$ *and* $1 \leq j \leq k_i$. *Then*

$$\Phi_{S_m}(t) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} A_{ij}\, \Phi_{Y_{ij}}(t) \quad (3.6)$$

Proof. The proof is a direct consequence of Corollary 1, knowing that for any RV X, $\Phi_X(t) = \int_{-\infty}^{+\infty} e^{tx} f_X(x)dx$.

Proposition 2. $\Phi_{S_m}(t) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} A_{ij} \frac{\alpha_i^j}{(\alpha_i - t)^j}$ *for* $t < min\{\vec{\alpha}\}$, $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$.

Proof. From Eq (3.6), $\Phi_{S_m}(t) = \sum_{i=1}^{n}\sum_{j=1}^{k_i} A_{ij}\, \Phi_{Y_{ij}}(t)$. However, the moment generating function of the Erlang Distribution $Y_{ij} \sim Erl(j, \alpha_i)$ is given by $\Phi_{Y_{ij}}(t) = \frac{\alpha_i^j}{(\alpha_i - t)^j}$ for $t < \alpha_i, i = 1,2,\dots,n$ see [13] and [8]. Hence $t < min\{\vec{\alpha}\}$, $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$. Thus, we obtain the result.

Corollary 6. *Let* $m \geq 2$ *and* $Y_{ij} \sim Erl(j, \alpha_i)$ *for* $1 \leq i \leq n$ *and* $1 \leq j \leq k_i$. *Then*

$$E[S_m^k] = \sum_{i=1}^{n}\sum_{j=1}^{k_i} A_{ij} E[Y_{ij}^k]. \tag{3.7}$$

Proof. We can directly obtain from Corollary 5 the result, where we have for the any RV X,

$$E[\mathrm{X}^k] = \frac{d^k \Phi_{\mathrm{X}}(t)}{dt^k}\Big|_{t=0}.$$

Proposition 3. *Let* $m \geq 2$ *and* k*be a positive integer. Then*

$$E[S_m^k] = \sum_{i=1}^{n}\sum_{j=1}^{k_i} A_{ij} \frac{j(j+1)\dots(j+k-1)}{\alpha_i^k}.$$

Proof. From Corollary 6, $E[S_m^k] = \sum_{i=1}^{n}\sum_{j=1}^{k_i} A_{ij} E[Y_{ij}^k]$. However, $Y_{ij} \sim Erl(j, \alpha_i)$ and the moment of Y_{ij} of order k is given by

$$E[Y_{ij}^k] = \frac{\Gamma(k+j)}{\alpha_i^k \Gamma(j)} \tag{3.8}$$

see [13]. Also $\frac{\Gamma(k+j)}{\Gamma(j)} = j(j+1)\dots(j+k-1)$. Therefore, we obtain the result.

In the next proposition, we use the idea of writing S_m as a sum of Erlang RV in Eq (2.1) to write the moment of S_n of order k in another new form.

Proposition 4. *Let* $m \geq 2$ *and* k*be a positive integer. Then*

$$E[S_m^k] = \sum_{E_k} k! \binom{k_1 + l_1 - 1}{l_1}\binom{k_2 + l_2 - 1}{l_2}\dots\binom{k_n + l_n - 1}{l_n} \cdot \frac{1}{\alpha_1^{l_1}} \cdot \frac{1}{\alpha_2^{l_2}} \cdots \frac{1}{\alpha_n^{l_n}},$$

where $E_k = \{(l_1, \dots, l_n)/0 \leq l_i \leq k;\ \sum_{i=1}^{n} l_i = k; 1 \leq i \leq n\}$.

Proof. We have from the definition that S_m can be written as a sum of Erlang Distribution, $Z_{k_i} \sim Erl(k_i, \alpha_i)$, $i = 1,2,..,n$ given in Eq (2.1). Moreover, $E[S_m^k] = E\left[(Z_{k_1} + Z_{k_2} + \dots + Z_{k_n})^k\right]$ and using multinomial expansion formula, we obtain that:

$E[S_m^k] = E\left[\sum_{E_k} \frac{k!}{l_1! l_2! \dots l_n!}\left(X_1^{l_1} X_2^{l_2} \dots X_n^{l_n}\right)\right]$. Knowing that expectation is linear and Z_{k_i}, $i = 1,2,\dots,n$ are independent having from Eq (3.8),$E[Z_k^l] = \frac{\Gamma(l+k)}{\alpha^l \Gamma(k)} = \binom{k+l-1}{l}\frac{l!}{\alpha^l}$, we obtain that

$$E[S_m^k] = \sum_{E_k} \frac{k!}{l_1!\, l_2! \ldots l_n!} E[Z_{k_1}^{l_1}].E[Z_{k_2}^{l_2}] \ldots E[Z_{k_n}^{l_n}]$$
$$= \sum_{E_k} k! \binom{k_1+l_1-1}{l_1}\binom{k_2+l_2-1}{l_2} \ldots \binom{k_n+l_n-1}{l_n} . \frac{1}{\alpha_1^{l_1}} . \frac{1}{\alpha_2^{l_2}} \ldots \frac{1}{\alpha_n^{l_n}}.$$

In the following two corollaries, we use Propositions 3 and 4 to formulate two important special cases: $k = 1$, Expectation, and $k = 2$. However, some Identities concerning A_{ij} have been established.

Corollary 7. $E[S_m] = \sum_{i=1}^{n} \sum_{j=1}^{k_i} A_{ij} \frac{j}{\alpha_i} = \sum_{i=1}^{n} \frac{k_i}{\alpha_i}$.

Proof. From Proposition 3, $E[S_m^k] = \sum_{i=1}^{n} \sum_{j=1}^{k_i} A_{ij} \frac{j(j+1)\ldots(j+k-1)}{\alpha_i^k}$. Set $k = 1$, then $E[S_m] = \sum_{i=1}^{n} \sum_{j=1}^{k_i} A_{ij} \frac{j}{\alpha_i}$. But $E[S_m] = \sum_{i=1}^{n} \frac{k_i}{\alpha_i}$. Thus we obtain the result.

Corollary 8. $E[S_m^2] = \sum_{i=1}^{n} \sum_{j=1}^{k_i} A_{ij} \frac{j(j+1)}{\alpha_i^2} = \sum_{i=1}^{n} \frac{k_i(k_i+1)}{\alpha_i^2} + 2 \sum_{1 \le i < j \le n}^{n} \frac{k_i k_j}{\alpha_i \alpha_j}$.

Proof. From Proposition 3, take $k = 2$, we obtain that $E[S_m^2] = \sum_{i=1}^{n} \sum_{j=1}^{k_i} A_{ij} \frac{j(j+1)}{\alpha_i^2}$. Moreover, from Proposition 4 we obtain that $E[S_m^2] = \sum_{E_k} 2! \binom{k_1+l_1-1}{l_1}\binom{k_2+l_2-1}{l_2} \ldots \binom{k_n+l_n-1}{l_n} . \frac{1}{\alpha_1^{l_1}} . \frac{1}{\alpha_2^{l_2}} \cdots \frac{1}{\alpha_n^{l_n}}$ $= \sum_{i=1}^{n} 2! \binom{k_i+1}{2} \frac{1}{\alpha_i^2} + \sum_{1 \le i < j \le n}^{n} 2! \binom{k_i}{1}\binom{k_j}{1} \frac{1}{\alpha_i \alpha_j} = \sum_{i=1}^{n} \frac{k_i(k_i+1)}{\alpha_i^2} + 2 \sum_{1 \le i < j \le n}^{n} \frac{k_i k_j}{\alpha_i \alpha_j}$.

3.4 Reliability and Hazard functions for the Hypoexponential RV

In this part we give a modified form of the reliability (survivor) and hazard (failure) functions.

The following corollary is a direct consequence of Corollary 4 and Proposition 1.

Corollary 9. *The reliability (survivor) function of the Hypoexponential RV* $Hypoexp(\vec{\alpha}, \vec{k})$, $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ *and* $\vec{k} = (k_1, k_2, \ldots, k_n)$ *is given by*

$$R_{S_m}(t) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} A_{ij} R_{Y_{ij}}(t) = \sum_{i=1}^{n} \sum_{j=1}^{k_i} A_{ij} \frac{\Gamma(j, \alpha_i t)}{(j-1)!} I_{(0,\infty)}(t),$$

where $\Gamma(j, \alpha_i t)$ is the upper incomplete Gamma function, see [14].

The following corollary is a direct consequence of Corollary 1 and 9.

Corollary 10. *The Hazard (failure) function for the Hypoexponential distribution* $Hypoexp(\vec{\alpha}, \vec{k})$, $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ *and* $\vec{k} = (k_1, k_2, \ldots, k_n)$ *is given by*

$$h_{S_m}(t) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{k_i} A_{ij} h_{Y_{ij}}(t) R_{Y_{ij}}(t)}{\sum_{i=1}^{n} \sum_{j=1}^{k_i} A_{ij} R_{Y_{ij}}(t)}.$$

4. Algorithm for Finding A_{ij}

Our method in finding A_{ij} presented in this section is a recursive algorithm. This method uses logarithmic properties and the Leibnitz's m^{th} derivative. In addition, we present an example showing the reliability and hazard functions.

Proposition 5. *Let* $1 \leq i \leq n$, $1 \leq j \leq k_i$. *Then* $A_{ij} = \frac{\prod_{i=1}^{n} \alpha_i^{k_i}}{\alpha_i^j (k_i - j)!} \lim_{s \to -\alpha_i} g_i^{(k_i - j)}(s)$.

Proof. We have from Eq (3.1) and (3.3), $A_{ij} = \left(\prod_{i=1}^{n} \alpha_i^{k_i}\right) \frac{c_{ij}}{\alpha_i^j}$ and $c_{ij} = \frac{1}{(k_i - j)!} \lim_{s \to -\alpha_i} g_i^{(k_i - j)}(s)$. respectively. Thus, we may write A_{ij} in the given form.

Theorem 3. *Let* $1 \leq i \leq n$, $1 \leq j \leq k_i$. *Then*

$$g_i^{(q)}(s) = \sum_{l=0}^{q-1} [\binom{q-1}{l} (-1)^{l+1} \left(\sum_{j=1, j \neq i}^{n} \frac{(l)! k_j}{(s + \alpha_j)^{l+1}} \right) g_i^{(q-l-1)}(s) \quad \text{for } q \geq 1.$$

Proof. We shall find the $(k_i - j)^{th}$ derivative of $g_i(s)$ in a recursive way. First, we apply logarithmic to $g_i(s) = \prod_{j=1, j \neq i}^{n} \frac{1}{(s + \alpha_j)^{k_j}}$, we obtain that $\ln g_i(s) = -\sum_{j=1, j \neq i}^{n} k_j \ln(s + \alpha_j)$.

Second, we differentiate the above equation with s, leads to $\frac{g_i'(s)}{g_i(s)} = -\sum_{j=1, j \neq i}^{n} \frac{k_j}{s + \alpha_j}$ and $g_i'(s) = -g_i(s) \sum_{j=1, j \neq i}^{n} \frac{k_j}{s + \alpha_j}$. Now, let $u_i = -\sum_{j=1, j \neq i}^{n} \frac{k_j}{s + \alpha_j}$ and $v_i = g_i(s)$, thus $g_i'(s) = u_i v_i$. By applying Leibnitz's q^{th} derivative to $u_i v_i$, we obtain

$$\left(g_i'(s)\right)^{(q)} = \sum_{l=0}^{q} [\binom{q}{l} u_i^{(l)}(s) v_i^{(q-l)}(s)].$$

However, $u_i^{(l)}(s) = (-1)^{l+1} \sum_{j=1, j \neq i}^{n} \frac{(l)! k_j}{(s + \alpha_j)^{l+1}}$ and $v_i^{(q-l)}(s) = g_i^{(q-l)}(s)$. Therefore,

$$g_i^{(q+1)}(s) = \sum_{l=0}^{q} [\binom{q}{l} (-1)^{l+1} \left(\sum_{j=1, j \neq i}^{n} \frac{(l)! k_j}{(s + \alpha_j)^{l+1}} \right) g_i^{(q-l)}(s)$$

Next, we give an application of a system consisting of independent exponential devises in stages.We show some curve illustrating our example.

Application. A system consists of 6 independent exponential devices in stages having parameters $\vec{\alpha} = (1,2,6)$ and $\vec{k} = (2,3,1)$. Then, the lifetime distribution of the system follows a Hypoexponential Distribution $Hypo\exp(\vec{\alpha}, \vec{k})$ having a PDF (Fig. 1) from Corollary 1, $f_{S_6}(t) = \sum_{i=1}^{3} \sum_{j=1}^{k_i} A_{ij} f_{Y_{ij}}(t) = -\frac{1}{200}(6e^{-6t}) + \frac{123}{8}(2e^{-2t}) - \frac{768}{25}(e^{-t}) + \frac{21}{4}(4te^{-2t}) + \frac{48}{5}(te^{-t}) + \frac{3}{2}(4t^2 e^{-2t})$.

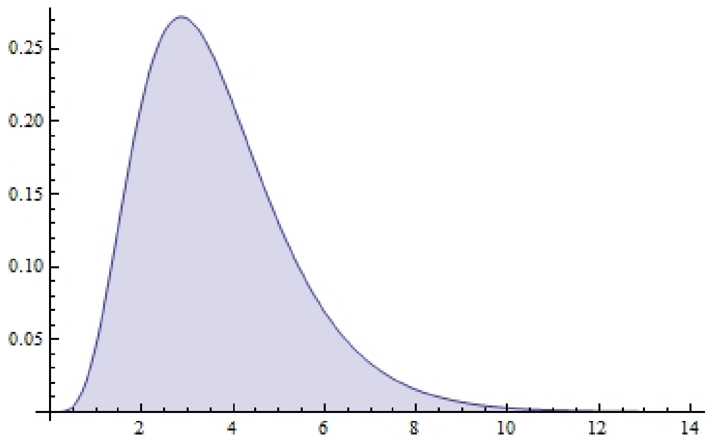

Fig. 1. lifetime distribution of the system

and having a Reliability (survival) function (Fig. 2) and Hazard Function (Fig. 3) from Corollaries 9 and 10,

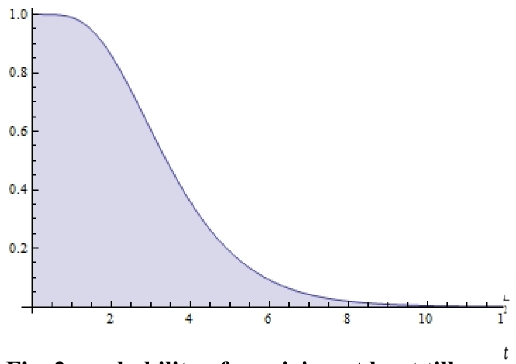

Fig. 2. probability of surviving at least till age

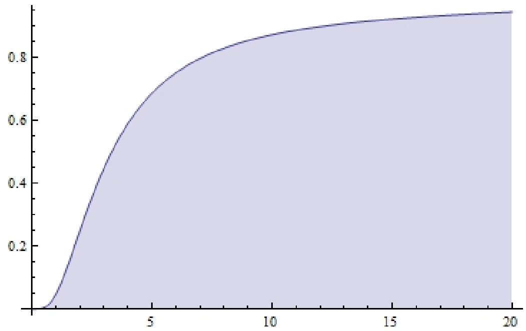

Fig. 3. System failure rate

5. Conclusion

A modified form of PDF, CDF, MGF, reliability function, hazard function and moment of order k for the general case of the Hypoexponential distribution were established. Moreover, some new related identities were given. The proof has been done by writing the PDF of the Hypoexponential distribution as a linear combination of the PDF of the known Erlang distribution and applying some linearity properties. Eventually, we find an algorithm for determining the coefficients of the linear combination, A_{ij}. The obtained identities containing A_{ij} maybe used in future work for finding a new method to compute A_{ij}.

Competing Interests

Authors have declared that no competing interests exist.

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