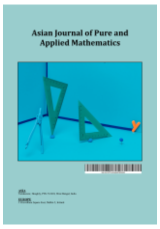

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Application of Collocation Methods for a Hybrid Block Scheme for the Solution of Volterra Intregral Equation of the Second Kind

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

In this paper, a class of three-step implicit second order hybrid block methods for the solutions of Volterra integral equation of the second kind has been developed, using the interpolation and collocation approach. The discrete block methods were recovered when the continuous block methods were evaluated at all step points. The block methods used to implement the main method guaranteed that each discrete scheme obtained from the simultaneous solution of the block has the same order of accuracy as the main continuous method. Hence, the new class of k-step methods gives high order of accuracy with very low error. The basic properties of the methods were investigated and the methods were found to be consistent, zero-stable and convergent.

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1 Introduction

Volterra integral equation is a special kind of integral equation which is classified into three: the first, second and third kind. In this research, we developed two off-grid points of hybrid block methods for the solution of second kind Volterra integral equation due to its characteristics and uniqueness. In this literature, the second kind Volterra integral equation (VIE) according to [1] is considered and is of the form:

$$y(x) = f(x) + \int_{x_0}^{x_n} \varphi\big(x,s)y(s)\big)ds \tag{1.0}$$

Where $f(x)$ is a given function and $\varphi(x,s)$ is called the kernel of integral equation. Volterra integral equation (VIE) appears especially when we try to transform an initial value problem into integral form, so that, the solution of the equation can be easily obtained than the original initial value problem [2]. Solving (1.0) is equivalent to solving the following initial value problem of the first order ordinary differential equations;

$$y'(x) = f'(x) + \varphi\big(x, y(x)\big),\ y(x_0) = f(x_0) \tag{1.1}$$

In the domains of engineering and applied research, the analytical and approximate techniques of solving the integral equations play a significant role. Some of these integral equations cannot be solved explicitly; hence approximation or numerical methods must frequently be used [3]. However, the collocation approach in this paper refers to a point at which the derivative is being evaluated.

In recent years, many strategies for resolving Volterra integral equations have been devised. Volterra-Fredholm integral equation solutions have recently been estimated using a variety of fundamental functions, including orthonormal bases and wavelets [4]. In an overview of integrators for the Volterra integral equations, methods like the Taylor method, transform method, method of variation, numerical techniques, direct quadrature method, Adomian decomposition method, homotopy Perturbation method, Galerkin and Finite Element method, collocation and spectral method, expansion method, and so on are discussed.

Lately, researchers such as: [5,6,7,8,9], and much recently, [10,11,12,13] suggested methods for solving Volterra integral equations.

2 Methodology of the Scheme

2.1 Derivation of the Proposed Method

Here, we derive three-step with two off-grid points of hybrid block method for the integration of Volterra Integral equation of second kind by carefully selecting $p = \frac{4}{3}$ **and** $q = \frac{5}{3}$ for $\boldsymbol{p}, \boldsymbol{q} \in [\mathbf{1}, \mathbf{2}]$

Let the approximate series solution and trigonometrically fitted function of the Eq. (1.0) be in the form of

$$z(x) = \sum_{j=0}^{3} \varphi_j x^j + \sum_{j=1}^{2} \lambda_j \sin x + \sum_{j=1}^{2} \lambda_j \cos x \tag{1.2}$$

Where $\emptyset_j$ and λ_j are the coefficients to be determined.

Consider the ordinary differential equation

$$z''= f\left(x, z, z'\right), z\left(a_0\right) = z_0 \;,\; z'\left(a\right) = z'_0 \tag{1.3}$$

Subject to

$$z\left(x\right) = y\left(x\right) - f\left(x\right) \tag{1.4}$$

The second derivative of Eq. (1.2) is given as;

$$z''\left(x\right) = \sum_{j=0}^{3} j\left(j-1\right)\phi_j x^{j-2} - \sum_{j=1}^{2} \lambda_j (\sin x) - \sum_{j=1}^{2} \lambda_j \left(\cos x\right) \tag{1.5}$$

Substituting Eq. (1.3) into (1.0) gives

$$g\left(x, z, z'\right) = \sum_{j=0}^{3} j\left(j-1\right)\varphi_j x^{j-2} - \sum_{j-1}^{2} \lambda_j \sin x - \sum_{j=1}^{2} \lambda_j \cos x \tag{1.6}$$

Interpolating (1.2) at $x_{n+k}, k = 1,2$ and collocating (1.5) at $x_{n+l,} l = \{0,1,\frac{4}{3},\frac{5}{3}2,3\}$ leads to the system of nonlinear equations written in the form

$$z_n\left(x\right) = X\left(x\right)A \tag{1.7}$$

$$\begin{bmatrix}
3 & 4x_n & -\frac{3}{2}x_n^2 & -\frac{1}{2}x_n^3 & \frac{17}{24}x_n^4 & \frac{11}{40}x_n^5 & -\frac{4}{45}x_n^6 & -\frac{8}{315}x_n^7 \\
3 & \left(4x_n + 4h\right) & -\frac{3}{2}\left(x_n + h\right)^2 & -\frac{1}{2}(x_n + h)^3 & \frac{17}{24}\left(x_n + h\right)^4 & \frac{11}{40}\left(x_n + h\right)^5 & -\frac{4}{45}\left(x_n + h\right)^6 & -\frac{8}{315}\left(x_n + h\right)^7 \\
0 & 0 & -3 & -3x_n & \frac{17}{2}x_n^2 & \frac{11}{2}x_n^3 & -\frac{8}{3}x_n^4 & -\frac{16}{15}x_n^5 \\
0 & 0 & -3 & -3x_n - h & \frac{17}{2}\left(x_n + h\right)^2 & \frac{11}{2}\left(x_n + h\right)^3 & -\frac{8}{3}\left(x_n + h\right)^4 & -\frac{16}{15}\left(x_n + h\right)^5 \\
0 & 0 & -3 & -3x_n - 2h & \frac{17}{2}\left(x_n + \frac{4}{3}h\right)^2 & \frac{11}{2}\left(x_n + \frac{4}{3}h\right)^3 & -\frac{8}{3}\left(x_n + \frac{4}{3}h\right)^4 & -\frac{16}{15}\left(x_n + \frac{4}{3}h\right)^5 \\
0 & 0 & -3 & -3x_n - 3h & \frac{17}{2}\left(x_n + \frac{5}{3}h\right)^2 & \frac{11}{2}\left(x_n + \frac{5}{3}h\right)^3 & -\frac{8}{3}\left(x_n + \frac{5}{3}h\right)^4 & -\frac{16}{15}\left(x_n + \frac{5}{3}h\right)^5 \\
0 & 0 & -3 & -3x_n - 6h & \frac{17}{2}\left(x_n + 2h\right)^2 & \frac{11}{2}\left(x_n + 2h\right)^3 & -\frac{8}{3}\left(x_n + 2h\right)^4 & -\frac{16}{15}\left(x_n + 2h\right)^5 \\
0 & 0 & -3 & -3x_n - 9h & \frac{17}{2}\left(x_n + 3h\right)^2 & \frac{11}{2}\left(x_n + 3h\right)^3 & -\frac{8}{3}\left(x_n + 3h\right)^4 & -\frac{16}{15}\left(x_n + 3h\right)^5
\end{bmatrix}
\begin{bmatrix} \psi_0 \\ \psi_1 \\ \phi_0 \\ \phi_1 \\ \phi_{\frac{4}{3}} \\ \phi_{\frac{5}{3}} \\ \phi_2 \\ \phi_3 \end{bmatrix}
=
\begin{bmatrix} z_n \\ z_{n+1} \\ g_n \\ g_{n+1} \\ g_{n+\frac{4}{3}} \\ g_{n+\frac{5}{3}} \\ g_{n+2} \\ g_{n+3} \end{bmatrix} \tag{1.8}$$

Using the Gaussian elimination method to solve the equation (1.8) gives the coefficients $\psi_0, \psi_1, \phi_0, \phi_1, \phi_{\frac{4}{3}}, \phi_{\frac{5}{3}}, \phi_2, \phi_3$ which are then substituted into (1.2) and simplified to give the implicit second derivative hybrid block method of the form;

$$z(x) = \sum_{i=0,1} \psi_i z_{n+i} + h^2 \left[\sum_{i=\frac{4}{3},\frac{5}{3}} \phi_i g_{n+i} + \sum_{i=0}^{3} \phi_i g_{n+i} \right] \tag{1.9}$$

Differentiating equation (1.9) once

$$z'(x) = \frac{1}{h} \sum_{i=0,1} \psi_i z_{n+i} + h \left[\sum_{i=\frac{4}{3},\frac{5}{3}} \phi_i g_{n+i} + \sum_{i=0}^{3} \phi_i g_{n+i} \right] \tag{2.0}$$

Where

$$\psi_0 = 1 - \frac{-x_n + x}{h} :$$

$$\psi_1 = \frac{-x_n + x}{h}$$

$$\phi_0 = -\frac{1261}{6300}\left(-x_n + x\right)h + \frac{1}{2}\left(-x_n + x\right)^2 - \frac{191}{360}\frac{\left(-x_n + x\right)^3}{h} + \frac{157}{480}\frac{\left(-x_n + x\right)^4}{h^2} - \frac{281}{2400}\frac{\left(-x_n + x\right)^5}{h^3} + \frac{9}{400}\frac{\left(-x_n + x\right)^6}{h^4} - \frac{1}{560}\frac{\left(-x_n + x\right)^7}{h^5}$$

$$\phi_1 = -\frac{2701}{1680}\left(-x_n + x\right)h + \frac{5\left(-x_n + x\right)^3}{h} - \frac{131}{24}\frac{\left(-x_n + x\right)^4}{h^2} + \frac{209}{80}\frac{\left(-x_n + x\right)^5}{h^3} - \frac{3}{5}\frac{\left(-x_n + x\right)^6}{h^4} + \frac{3}{56}\frac{\left(-x_n + x\right)^7}{h^5}$$

$$\phi_{\frac{4}{3}} = \frac{486}{175}\left(-x_n + x\right)h - \frac{81}{8}\frac{\left(-x_n + x\right)^3}{h} + \frac{1971}{160}\frac{\left(-x_n + x\right)^4}{h^2} - \frac{5103}{800}\frac{\left(-x_n + x\right)^5}{h^3} + \frac{621}{400}\frac{\left(-x_n + x\right)^6}{h^4} - \frac{81}{560}\frac{\left(-x_n + x\right)^7}{h^5}$$

$$\phi_{\frac{5}{3}} = -\frac{11583}{5600}\left(-x_n + x\right)h + \frac{81}{10}\frac{\left(-x_n + x\right)^3}{h} - \frac{837}{80}\frac{\left(-x_n + x\right)^4}{h^2} + \frac{4617}{800}\frac{\left(-x_n + x\right)^5}{h^3}$$
$$-\frac{297}{200}\frac{\left(-x_n + x\right)^6}{h^4} + \frac{81}{560}\frac{\left(-x_n + x\right)^7}{h^5}$$

$$\phi_2 = \frac{257}{420}\left(-x_n + x\right)h - \frac{5}{2}\frac{\left(-x_n + x\right)^3}{h} + \frac{161}{48}\frac{\left(-x_n + x\right)^4}{h^2} - \frac{31}{16}\frac{\left(-x_n + x\right)^5}{h^3}$$
$$+\frac{21}{40}\frac{\left(-x_n + x\right)^6}{h^4}$$
$$-\frac{3}{56}\frac{\left(-x_n + x\right)^7}{h^5}$$

$$\phi_3 = -\frac{643}{50400}\left(-x_n + x\right)h + \frac{1}{18}\frac{\left(-x_n + x\right)^3}{h} - \frac{19}{240}\frac{\left(-x_n + x\right)^4}{h^2} + \frac{119}{2400}\frac{\left(-x_n + x\right)^5}{h^3}$$
$$-\frac{3}{200}\frac{\left(-x_n + x\right)^6}{h^4}\ \frac{1}{560}\frac{\left(-x_n + x\right)^7}{h^5}$$

Evaluating (1.2) at non-interpolating points $x = x_{n+\frac{4}{3}}, x_{n+\frac{5}{3}}, x_{n+2}, x_{n+3}$ yields

$$z_{n+\frac{4}{3}} = -\frac{1}{3}z_n + \frac{4}{3}z_{n+1} + \frac{973}{54675}h^2 g_n + \frac{7237}{14580}h^2 g_{n+1} - \frac{136}{225}h^2 g_{n+\frac{4}{3}} + \frac{2351}{5400}h^2 g_{n+\frac{5}{3}}$$
$$-\frac{457}{3645}h^2 g_{n+2} + \frac{1099}{437400}h^2 g_{n+3}$$

$$z_{n+\frac{5}{3}} = -\frac{2}{3}z_n + \frac{5}{3}z_{n+1} + \frac{6223}{174960}h^2 g_n + \frac{2923}{2916}h^2 g_{n+1} - \frac{2413}{2160}h^2 g_{n+\frac{4}{3}} + \frac{317}{360}h^2 g_{n+\frac{5}{3}}$$
$$-\frac{1463}{5832}h^2 g_{n+2} + \frac{439}{87480}h^2 g_{n+3}$$

$$z_{n+2} = -z_n + 2z_{n+1} + \frac{4}{75}h^2 g_n + \frac{181}{120}h^2 g_{n+1} - \frac{81}{50}h^2 g_{n+\frac{4}{3}} + \frac{567}{400}h^2 g_{n+\frac{5}{3}} - \frac{11}{30}h^2 g_{n+2}$$
$$+\frac{3}{400}h^2 g_{n+3}$$

$$z_{n+3} = -2z_n + 3z_{n+1} + \frac{137}{1200}h^2 g_n + \frac{53}{20}h^2 g_{n+1} - \frac{729}{400}h^2 g_{n+\frac{4}{3}} + \frac{243}{200}h^2 g_{n+\frac{5}{3}} + \frac{31}{40}h^2 g_{n+2} + \frac{41}{600}h^2 g_{n+3} \quad (2.1)$$

We then evaluate (2.0) at all points we have,

$$hz'_n = -z_n + z_{n+1} - \frac{1261}{6300}h^2 g_n - \frac{2701}{1680}h^2 g_{n+1} + \frac{486}{175}h^2 g_{n+\frac{4}{3}} - \frac{11583}{5600}h^2 g_{n+\frac{5}{3}} + \frac{257}{420}h^2 g_{n+2} - \frac{643}{50400}h^2 g_{n+3}$$

$$hz'_{n+1} = -z_n + z_{n+1} + \frac{2701}{50400}h^2 g_n + \frac{391}{280}h^2 g_{n+1} - \frac{10719}{5600}h^2 g_{n+\frac{4}{3}} + \frac{3753}{2800}h^2 g_{n+\frac{5}{3}} - \frac{43}{112}h^2 g_{n+2} + \frac{193}{25200}h^2 g_{n+3}$$

$$hz'_{n+\frac{4}{3}} = -z_n + z_{n+1} + \frac{27163}{510300}h^2 g_n + \frac{41543}{27216}h^2 g_{n+1} - \frac{2612}{1575}h^2 g_{n+\frac{4}{3}} + \frac{64201}{50400}h^2 g_{n+\frac{5}{3}} - \frac{12559}{34020}h^2 g_{n+2} + \frac{30349}{4082400}h^2 g_{n+3}$$

$$hz'_{n+\frac{5}{3}} = -z_n + z_{n+1} + \frac{217997}{4082400}h^2 g_n + \frac{102797}{68040}h^2 g_{n+1} - \frac{74407}{50400}h^2 g_{n+\frac{4}{3}} + \frac{36689}{25200}h^2 g_{n+\frac{5}{3}} - \frac{52357}{136080}h^2 g_{n+2} + \frac{15521}{2041200}h^2 g_{n+3}$$

$$hz'_{n+2} = -z_n + z_{n+1} + \frac{67}{1260}h^2 g_n + \frac{2563}{1680}h^2 g_{n+1} - \frac{54}{35}h^2 g_{n+\frac{4}{3}} + \frac{1917}{1120}h^2 g_{n+\frac{5}{3}} - \frac{107}{420}h^2 g_{n+2} + \frac{73}{10080}h^2 g_{n+3}$$

$$hz'_{n+3} = -z_n + z_{n+1} + \frac{3709}{50400}h^2 g_n + \frac{89}{168}h^2 g_{n+1} + \frac{10449}{5600}h^2 g_{n+\frac{4}{3}} - \frac{8343}{2800}h^2 g_{n+\frac{5}{3}}$$

$$+ \frac{4619}{1680}h^2 g_{n+2}$$

$$+ \frac{6577}{25200}h^2 g_{n+3} \tag{2.2}$$

This gives the following matrix equation in the form

$$A_1 Z = A_2 R_1 + B_1 R_2 + B_2 R_3 \tag{2.3}$$

$$A_1 = \begin{bmatrix} -\frac{4}{3} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{5}{3} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{h} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{h} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{1}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{h} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, Z = \begin{bmatrix} z_{n+1} \\ z_{n+\frac{4}{3}} \\ z_{n+\frac{5}{3}} \\ z_{n+2} \\ z_{n+3} \\ z'_{n+1} \\ z'_{n+\frac{4}{3}} \\ z'_{n+\frac{5}{3}} \\ z'_{n+2} \\ z'_{n+3} \end{bmatrix}, A_2 = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{2}{3} & 0 \\ 1 & 0 \\ 2 & 0 \\ \frac{1}{h} & 1 \\ \frac{1}{h} & 0 \\ \frac{1}{h} & 0 \\ \frac{1}{h} & 0 \\ \frac{1}{h} & 0 \\ \frac{1}{h} & 0 \end{bmatrix}, R_1 = \begin{bmatrix} z_n \\ z'_n \end{bmatrix}, B_1 = \begin{bmatrix} -\frac{973}{54675}h^2 \\ \frac{6223}{174960}h^2 \\ \frac{4}{75}h^2 \\ \frac{137}{1200}h^2 \\ -\frac{1261}{6300}h \\ \frac{2701}{50400}h \\ \frac{27163}{510300}h \\ \frac{217997}{4082400}h \\ \frac{67}{1260}h \\ \frac{3709}{5040}h \end{bmatrix}, R_2 = \begin{bmatrix} g_n \end{bmatrix},$$

$$B_2 = \begin{bmatrix} \frac{7237}{14580}h^2 & -\frac{136}{225}h^2 & \frac{2351}{5400}h^2 & -\frac{457}{3647}h^2 & \frac{1099}{437400}h^2 \\ \frac{2923}{2916}h^2 & -\frac{2413}{2160}h^2 & \frac{317}{360}h^2 & -\frac{1463}{5832}h^2 & \frac{439}{87480}h^2 \\ \frac{181}{120}h^2 & -\frac{81}{50}h^2 & \frac{567}{400}h^2 & -\frac{11}{30}h^2 & \frac{3}{400}h^2 \\ \frac{53}{20}h^2 & -\frac{729}{400}h^2 & \frac{243}{200}h^2 & \frac{31}{40}h^2 & \frac{41}{600}h^2 \\ -\frac{2701}{1680}h^2 & \frac{486}{155}h^2 & -\frac{11583}{6500}h^2 & \frac{275}{420}h^2 & -\frac{643}{56400}h^2 \\ \frac{391}{280}h^2 & -\frac{10719}{5600}h^2 & \frac{3753}{2800}h^2 & -\frac{12559}{34020}h^2 & \frac{30349}{4082400}h^2 \\ \frac{41543}{27216}h^2 & -\frac{2612}{1575}h^2 & \frac{64201}{50400}h^2 & -\frac{12559}{34020}h^2 & \frac{30349}{4082400}h^2 \\ \frac{102797}{68040}h^2 & -\frac{74407}{50400}h^2 & \frac{36689}{25200}h^2 & -\frac{52357}{136080}h^2 & \frac{15521}{2041200}h^2 \\ \frac{2563}{1680}h^2 & -\frac{54}{35}h^2 & \frac{1917}{1120}h^2 & \frac{107}{420}h^2 & \frac{73}{10080}h^2 \\ \frac{89}{168}h^2 & \frac{10449}{5600}h^2 & -\frac{8343}{2800}h^2 & \frac{4619}{1680}h^2 & \frac{6577}{25200}h^2 \end{bmatrix}, R_3 = \begin{bmatrix} g_1 \\ g_{\frac{4}{3}} \\ g_{\frac{5}{3}} \\ g_2 \\ g_3 \end{bmatrix} \tag{2.4}$$

Substituting (2.2) into (2.3) and multiply by the inverse A_1 gives the hybrid block in the form;

$$
\begin{bmatrix}
1&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&1
\end{bmatrix}
\begin{bmatrix}
z_{n+1}\\ z_{n+\frac{4}{3}}\\ z_{n+\frac{5}{3}}\\ z_{n+2}\\ z_{n+3}\\ z'_{n+1}\\ z'_{n+\frac{4}{3}}\\ z'_{n+\frac{5}{3}}\\ z'_{n+2}\\ z'_{n+3}
\end{bmatrix}
=
\begin{bmatrix}
1 & h\\
1 & \frac{4}{3}h\\
1 & \frac{5}{3}h\\
1 & 2h\\
1 & 3h\\
0 & 1\\
0 & 1\\
0 & 1\\
0 & 1\\
0 & 1
\end{bmatrix}
\begin{bmatrix} z_n\\ z'_n \end{bmatrix}
+
\begin{bmatrix}
\frac{1261}{6300}h^2\\
\frac{108952}{3827251}h^2\\
\frac{90425}{244944}h^2\\
\frac{1429}{3150}h^2\\
\frac{2001}{2800}h^2\\
\frac{203}{800}h\\
\frac{4618}{18225}h\\
\frac{5915}{23328}h\\
\frac{19}{75}h\\
\frac{219}{800}h
\end{bmatrix}
\begin{bmatrix} g_n \end{bmatrix}
+
\begin{bmatrix}
\frac{2701}{1680}h^2 & -\frac{486}{175}h^2 & \frac{11583}{5600}h^2 & -\frac{257}{420}h^2 & \frac{643}{50400}h^2\\
\frac{13472}{5103}h^2 & -\frac{6780}{1575}h^2 & \frac{15088}{4725}h^2 & -\frac{24016}{25515}h^2 & \frac{7472}{382725}h^2\\
\frac{300625}{81648}h^2 & -\frac{17375}{3024}h^2 & \frac{8725}{2016}h^2 & -\frac{51875}{40825}h^2 & \frac{12875}{489888}h^2\\
\frac{496}{105}h^2 & -\frac{2511}{350}h^2 & \frac{972}{175}h^2 & -\frac{167}{105}h^2 & \frac{52}{1575}h^2\\
\frac{837}{112}h^2 & -\frac{28431}{2800}h^2 & \frac{41553}{5600}h^2 & -\frac{297}{280}h^2 & \frac{597}{5600}h^2\\
\frac{721}{240}h & -\frac{3753}{800}h & \frac{2727}{800}h & -\frac{239}{240}h & \frac{49}{2400}h\\
\frac{3808}{1215}h & -\frac{1225}{288}h & \frac{1015}{288}h & -\frac{3875}{3888}h & \frac{475}{23328}h\\
\frac{12125}{3888}h & -\frac{1225}{288}h & \frac{1015}{288}h & -\frac{3875}{3888}h & \frac{475}{23328}h\\
\frac{47}{15}h & -\frac{108}{25}h & \frac{189}{50}h & -\frac{13}{15}h & \frac{1}{50}h\\
\frac{171}{80}h & -\frac{729}{800}h & -\frac{729}{800}h & \frac{171}{80}h & \frac{219}{800}h
\end{bmatrix}
\begin{bmatrix} g1\\ g_{\frac{4}{3}}\\ g_{\frac{5}{3}}\\ g_2\\ g_3 \end{bmatrix}
\tag{2.5}
$$

By putting (1.4) in (2.5) yields the Volterra discrete scheme used in block form as

$$
\begin{aligned}
y_{n+1} - f_{n+1} = y_n - f_n + h\left(y'_n - f'_n\right) + \frac{1261}{6300}h^2 g_n + \frac{2701}{1680}h^2 g_{n+1}\\
-\frac{486}{175}h^2 g_{n+\frac{4}{3}} + \frac{11583}{5600}h^2 g_{n+\frac{5}{3}}\\
-\frac{257}{420}h^2 g_{n+2} + \frac{643}{50400}h^2 g_{n+3}
\end{aligned}
$$

$$
\begin{aligned}
y_{n+\frac{4}{3}} - f_{n+\frac{4}{3}} = y_n - f_n + \frac{4}{3}\left(y'_n - f'_n\right) + \frac{108952}{382725}h^2 g_n + \frac{13472}{5103}h^2 g_{n+1} - \frac{6784}{1575}h^2 g_{n+\frac{4}{3}}\\
+ \frac{15088}{4725}h^2 g_{n+\frac{5}{3}} - \frac{24016}{25515}h^2 g_{n+2} + \frac{7472}{382725}h^2 g_{n+3}
\end{aligned}
$$

$$
\begin{aligned}
y_{n+\frac{5}{3}} - f_{n+\frac{5}{3}} = y_n - f_n + \frac{5}{3}\left(y'_n - f'_n\right) + \frac{90425}{244944}h^2 g_n + \frac{300625}{81648}h^2 g_{n+1} - \frac{17375}{3024}h^2 g_{n+\frac{4}{3}}\\
+ \frac{8725}{2016}h^2 g_{n+\frac{5}{3}} - \frac{51875}{40824}h^2 g_{n+2} + \frac{12875}{489888}h^2 g_{n+3}
\end{aligned}
$$

$$y_{n+2} - f_{n+2} = y_n - f_n + 2\left(y'_n - f_n\right) + \frac{1429}{3150} h^2 g_n + \frac{496}{105} h^2 g_{n+1} - \frac{2511}{350} h^2 g_{n+\frac{4}{3}} + \frac{972}{175} h^2 g_{n+\frac{5}{3}} - \frac{167}{105} h^2 g_{n+2} + \frac{52}{1575} h^2 g_{n+3}$$

$$y_{n+3} - f_{n+3} = y_n - f_n + 3\left(y'_n - f_n\right) + \frac{2001}{2800} h^2 g_n + \frac{837}{112} h^2 g_{n+1} - \frac{28431}{2800} h^2 g_{n+\frac{4}{3}} + \frac{41553}{5600} h^2 g_{n+\frac{5}{3}} - \frac{297}{280} h^2 g_{n+2} + \frac{597}{5600} h^2 g_{n+3}$$

$$y'_{n+1} - f_{n+1} = \left(y'_n - f_n\right) + \frac{203}{800} h g_n + \frac{721}{240} h g_{n+1} - \frac{3753}{800} h g_{n+\frac{4}{3}} + \frac{2727}{800} h g_{n+\frac{5}{3}} - \frac{239}{240} h g_{n+2} + \frac{49}{2400} h g_{n+3}$$

$$y'_{n+\frac{4}{3}} - f_{n+\frac{4}{3}} = \left(y'_n - f_n\right) + \frac{4618}{18225} h g_n + \frac{3808}{1215} h g_{n+1} - \frac{998}{225} h g_{n+\frac{4}{3}} + \frac{752}{225} h g_{n+\frac{5}{3}} - \frac{1192}{1215} h g_{n+2} + \frac{368}{18225} h g_{n+3}$$

$$y'_{n+\frac{5}{3}} - f_{n+\frac{5}{3}} = \left(y'_n - f_n\right) + \frac{5915}{23328} h g_n + \frac{12125}{3888} h g_{n+1} - \frac{1225}{288} h g_{n+\frac{4}{3}} + \frac{1015}{288} h g_{n+\frac{5}{3}} - \frac{3875}{3888} h g_{n+2} + \frac{475}{23328} h g_{n+3}$$

$$y'_{n+2} - f_{n+2} = \left(y'_n - f_n\right) + \frac{19}{75} h g_n + \frac{47}{15} h g_{n+1} - \frac{108}{25} h g_{n+\frac{4}{3}} + \frac{189}{50} h g_{n+\frac{5}{3}} - \frac{13}{15} h g_{n+2} + \frac{1}{50} h g_{n+3} :$$

$$y'_{n+3} - f_{n+3} = \left(y'_n - f_n\right) + \frac{219}{800} h g_n + \frac{171}{80} h g_{n+1} - \frac{729}{800} h g_{n+\frac{4}{3}} - \frac{729}{800} h g_{n+\frac{5}{3}} + \frac{171}{80} h g_{n+2} + \frac{219}{800} h g_{n+3} \qquad (2.6)$$

3. Analysis of Three-Step Implicit Second Derivative Hybrid Method with Two Off-Grid Points

The basic properties of the new implicit second derivative hybrid methods are analyzed in this section to establish their validity. The properties to be considered are: order, error constant, consistency, zero-stability and convergence. To determine these properties, we adopted the procedure given in Lambert (1991).

2.2 Order and Error Constant of the Proposed Method

Let the linear difference operator ℓ associated with the new method (16) be defined as

$$L[y(x;h)] = \sum_{j=0}^{k} \alpha_j y(x+jh) - h^2 \sum_{j=0}^{k} (\beta_j y''(x+jh)) \qquad (2.7)$$

Where $y(x)$ is an arbitrary test function continuously differential on [a, b]. Expanding $y(x+jh)$, $y'(x+jh$ and $y''(x+jh$ of (16) in Taylor series in the form:

$$\ell[y(x);h] = \bar{c}_0 y(x) + \bar{c}_1 h y'(x) + \bar{c}_2 h^2 y''(x) + \cdots + \bar{c}_{p+2} h^{p+2} y^{(p+2)}(x) + \cdots \qquad (2.8)$$

Similarly,

The new implicit second derivative hybrid method (3.10) is of order p if,

$$\ell[y(x);h] = 0(h^{p+2}), \bar{c}_0 = \bar{c}_1 = \bar{c}_{2=} \dots = \bar{c}_{p+1} = 0, \bar{c}_{p+2} \neq 0$$

Therefore, the principal local truncation error $x_n + k$ is then defined to be

$$\bar{c}_{p+2} h^{p+2} y^{(p+2)}(xn)$$

Where

$$\vdots$$

$$C_0 = \sum_{j=0}^{k} \alpha_j$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k} \beta_j$$

$$C_2 = \frac{1}{2}\sum_{j=1}^{k} j^2\alpha_j - \sum_{j=0}^{k} j\beta_j$$

$$\vdots$$

$$C_q = \frac{1}{q!}\sum_{j=1}^{k} j^q \alpha_j - \frac{1}{(q-1)!}\sum_{j=0}^{k} j^{q-1}\beta_j , q = 2,3,4,5, \cdots \qquad (2.9)$$

$$\begin{bmatrix}
\sum_{j=0}^{\infty}\frac{(h)^j}{j!}(z_n)^j - z_n - hz'_n - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!} g_n^{j+2}\left[\frac{1261}{6300}+\frac{2701}{1680}(1)-\frac{486}{175}\left(\frac{4}{3}\right)+\frac{11583}{5600}\left(\frac{5}{3}\right)-\frac{257}{420}(2)+\frac{643}{50400}(3)\right] \\
\sum_{j=0}^{\infty}\frac{\left(\frac{4}{3}h\right)^j}{j!}(z_n)^j - z_n - \frac{4}{3}hz'_n - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!} g_n^{j+2}\left[\frac{108952}{382725}+\frac{13472}{5103}(1)-\frac{6784}{1575}\left(\frac{4}{3}\right)+\frac{15088}{4725}\left(\frac{5}{3}\right)-\frac{24016}{25515}(2)+\frac{7472}{382725}(3)\right] \\
\sum_{j=0}^{\infty}\frac{\left(\frac{5}{3}h\right)^j}{j!}(z_n)^j - z_n - \frac{5}{3}hz'_n - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!} g_n^{j+2}\left[\frac{190425}{244944}+\frac{300625}{81648}(1)-\frac{17375}{3024}\left(\frac{4}{3}\right)+\frac{8725}{2016}\left(\frac{5}{3}\right)-\frac{51875}{40824}(2)+\frac{12875}{382725}(3)\right] \\
\sum_{j=0}^{\infty}\frac{(2h)^j}{j!}(z_n)^j - z_n - 2hz'_n - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!} g_n^{j+2}\left[\frac{1429}{3150}+\frac{496}{105}(1)-\frac{2511}{350}\left(\frac{4}{3}\right)+\frac{972}{175}\left(\frac{5}{3}\right)-\frac{167}{105}(2)+\frac{52}{1575}(3)\right] \\
\sum_{j=0}^{\infty}\frac{(3h)^j}{j!}(z_n)^j - z_n - 3hz'_n - \sum_{j=0}^{\infty}\frac{h^{j+2}}{j!} g_n^{j+2}\left[\frac{2001}{280}+\frac{837}{112}\left(\frac{1}{3}\right)-\frac{28431}{2800}\left(\frac{2}{3}\right)+\frac{41553}{5600}(1)-\frac{297}{280}(2)+\frac{597}{5600}(3)\right] \\
\sum_{j=0}^{\infty}\frac{(1)^j}{j!}(z_n)^j - z'_n - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!} g_n^{j+2}\left[\frac{203}{800}+\frac{721}{240}(1)-\frac{3753}{800}\left(\frac{4}{3}\right)+\frac{2727}{800}\left(\frac{5}{3}\right)-\frac{239}{240}(2)+\frac{49}{2400}(3)\right] \\
\sum_{j=0}^{\infty}\frac{\left(\frac{4}{3}\right)^j}{j!}(z_n)^j - z'_n - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!} g_n^{j+2}\left[\frac{4618}{18225}+\frac{3808}{1215}(1)-\frac{998}{225}\left(\frac{4}{3}\right)+\frac{752}{225}\left(\frac{5}{3}\right)-\frac{1192}{1215}(2)+\frac{368}{18225}(3)\right] \\
\sum_{j=0}^{\infty}\frac{\left(\frac{5}{3}\right)^j}{j!}(z_n)^j - z'_n - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!} g_n^{j+2}\left[\frac{5915}{23328}+\frac{12125}{3888}(1)-\frac{1225}{288}\left(\frac{4}{3}\right)+\frac{1015}{288}\left(\frac{5}{3}\right)-\frac{3875}{3888}(2)+\frac{475}{23328}(3)\right] \\
\sum_{j=0}^{\infty}\frac{(2)^j}{j!}(z_n)^j - z'_n - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!} g_n^{j+2}\left[\frac{19}{75}+\frac{47}{15}(1)-\frac{108}{25}\left(\frac{4}{3}\right)+\frac{189}{50}\left(\frac{5}{3}\right)-\frac{13}{15}(2)+\frac{1}{50}(3)\right] \\
\sum_{j=0}^{\infty}\frac{(3)^j}{j!}(z_n)^j - z'_n - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!} g_n^{j+2}\left[\frac{219}{800}+\frac{171}{80}(1)-\frac{729}{800}\left(\frac{4}{3}\right)-\frac{729}{800}\left(\frac{5}{3}\right)+\frac{171}{80}(2)+\frac{219}{800}(3)\right]
\end{bmatrix} \qquad (3.0)$$

$C_0 = C_1 = C_2 = C_3 = C_4 = C5 = C_6 . . . = C_7 = 0$ and

$$c_8 = \left[-\frac{1153}{1814400}, -\frac{29912}{31000725}, -\frac{205375}{158723712}, -\frac{23}{14175}, -\frac{199}{67200}, -\frac{2701}{2721600}, -\frac{3392}{3444525}, -\frac{8725}{8817984}, -\frac{3767}{28350}, -\frac{199}{100800} \right]$$

C_8 Is the truncation error, $c_{p+2} = 7$, in which its order is $p = (5,5,5,5,5,5,5,5,5,5)^T$

3 Results and Discussion

The method adopted for the implementation is such that the entire discrete scheme obtained from the continuous schemes which have the same order of accuracy with low error constant for values of h are combined as simultaneous integrators.

The absolute errors calculated in the code are define by

$$ERR = |Yc - Yex|$$

Where Yex is the exact solution, Yc is the computed result and ERR is the absolute error.

All computations were carried out using MALPE 2015 and MATLAB 2013 version, the computer codes are simply written and requiring no previous knowledge of programming before it can be used.

3.1 Numerical Examples

In order to study the efficiency of the developed method, two numerical examples are presented. The class of continuous implicit hybrid k-step methods: Case 1, Case 2 and Case 3 were applied to solve the following Volterra integral problems

Problem 1.

Consider the second kind linear volterra integral equation

$$X(t) = t^2 + \int_0^x (t - s)x_1(s)ds$$

With exact solution

$$X(t) = 2cosht - 2, h = 0.1$$

(Maturi *et al*. (2014))

Problem 2.

Consider the second kind linear volterra integral equation

$$U(x) = 1 + x + \int_0^x (x - t)U(t)dt$$

With exact solution:

$$U(x) = e^x, h = 0.1$$

Source: Shoukralla and Ahmed (2020)

Table 1. Showing the exact solution, computed results from the propose method, error in the new scheme and error in Maturi et al for problem 1

X	Exact Solution	New Scheme (computed result)	Err in New Scheme	Err Maturi *et al* (2014)
0.1	0.01000833611160719800	0.01000833612444900235	1.2841e-11	1.0e-05
0.2	0.04013351123815169260	0.04013351127106888506	3.2917e-11	3.0e-05
0.3	0.09067702825772097000	0.09067702831817961927	6.0458e-11	8.0e-05
0.4	0.16214474367690961860	0.16214474379236288043	1.1545e-10	1.40e-04
0.5	0.25525193041276157040	0.25525193059208270052	1.79321e-10	2.20e-04
0.6	0.37093043648453540760	0.37093043673733442566	2.5279e-10	3.20e-04
0.7	0.51033801126188603640	0.51033801162114006499	3.5925e-10	4.40e-04
0.8	0.67486989260968919600	0.67486989308802405620	4.7833e-10	5.90e-04
0.9	0.86617277089754877560	0.86617277150895447806	6.11405e-10	7.70e-04
1.0	1.08616126963048755700	1.08616127041789008540	7.87402e-10	9.80e-04

Table 2. Showing the exact solution and computed results from the propose methods for problem 2

X	Exact Solution	New Scheme (computed result)	Err in New Scheme	Erro in Shoukralla and ahmed (2020)
0.1	1.1051709180756476248	1.10517091808291651920	7.2689e-12	1.4089e-09
0.2	1.2214027581601698339	1.22140275817880999460	1.8640e-11	9.1493e-08
0.3	1.3498588075760031040	1.34985880761035249250	3.4349-11	1.0576e-05
0.4	1.4918246976412703178	1.49182469770900733720	6.7737e-11	6.0309e-06
0.5	1.6487212707001281468	1.64872127080737015330	1.0724e-10	2.3354e-05
0.6	1.8221188003905089749	1.82211880054393309260	1.5342e-10	7.08004e-05
0.7	2.0137527074704765216	2.01375270769501756400	2.2454e-10	1.8129e-04
0.8	2.2255409284924676046	2.22554092879771530440	3.0525e-10	4.1026e-04
0.9	2.4596031111569496638	2.45960311155352339510	3.9657e-10	8.4486e-04
1	2.7182818284590452354	2.71828182898249978340	5.2345e-10	1.6151e-03

4 Conclusion

We observed from the above table 1 and 2 that, the numerical results obtained are more favorably, converged quickly and produced better approximations in comparison to other existing methods.

In the tables above, the property of our proposed method showed that the scheme is consistent and also convergent when we compare our results obtained with [7] and [10]

Competing Interests

Authors have declared that no competing interests exist.

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