World Journal of Mechanics, 2012, 2, 98-112
doi:10.4236/wjm.2012.22012 Published Online April 2012 (http://www.SciRP.org/journal/wjm)

3-D Exact Vibration Analysis of a Generalized Thermoelastic Hollow Sphere with Matrix Frobenius Method

Jagan Nath Sharma, Nivedita Sharma
Department of Mathematics, National Institute of Technology, Hamirpur, India
Email: jns@nitham.ac.in, niveditanithmr@gmail.com

Received December 28, 2011; revised January 29, 2012; accepted February 15, 2012

ABSTRACT

This paper presents exact free vibration analysis of stress free (or rigidly fixed), thermally insulated (or isothermal), transradially isotropic thermoelastic hollow sphere in context of generalized (non-classical) theory of thermoelasticity. The basic governing equations of linear generalized thermoelastic transradially isotropic hollow sphere have been uncoupled and simplified with the help of potential functions by using the Helmholtz decomposition theorem. Upon using it the coupled system of equations reduced to ordinary differential equations in radial coordinate. Matrix Frobenius method of extended series has been used to investigate the motion along the radial coordinate. The secular equations for the existence of possible modes of vibrations in the considered sphere are derived. The special cases of spheroidal (S-mode) and toroidal (T-mode) vibrations of a hollow sphere have also been deduced and discussed. The toroidal motion gets decoupled from the spheroidal one and remains independent of the both, thermal variations and thermal relaxation time. In order to illustrate the analytic results, the numerical solution of the secular equation which governs spheroidal motion (S-modes) is carried out to compute lowest frequencies of vibrational modes in case of classical (CT) and non-classical (LS, GL) theories of thermoelasticity with the help of MATLAB programming for the generalized hollow sphere of helium and magnesium materials. The computer simulated results have been presented graphically showing lowest frequency and dissipation factor. The analysis may find applications in engineering industries where spherical structures are in frequent use.

Keywords: Thermal Relaxation; Matrix Frobenius Method; Toroidal; Poloidal; Hollow Sphere

1. Introduction

The theory of thermoelasticity is well established, Nowacki [1]. The governing field equations in classical dynamic coupled thermoelasticity (CT) are wave-type (hyperbolic) equations of motion and a diffusion-type (parabolic) equation of heat conduction. Therefore, it is seen that part of the solution of energy equation extends to infinity, implying that if a homogeneous isotropic elastic medium is subjected to thermal or mechanical disturbances, the effect of temperature and displacement fields is felt at an infinite distance from the source of disturbance. This shows that part of disturbance has an infinite velocity of propagation, which is physically impossible. With this drawback in mind, Lord and Shulman [2], Green and Lindsay [3], modified the Fourier law of heat conduction and constitutive relations so as to get a hyperbolic equation for heat conduction. These works include the time needed for the acceleration of heat flow and take into account the coupling between temperature and strain fields for isotropic materials. Dhaliwal and Sherief [4] extended the generalized thermoelasticity [2] to anisotropic elastic bodies. A wave-like thermal disturbance is referred as "second sound" by Chandrasekharaiah [5]. These theories are also supported by experiments of Ackerman *et al*. [6] that exhibiting the actual occurrence of second sound at low temperatures and small intervals of time. The investigators Singh and Sharma [7] studied the propagation of plane harmonic waves in homogeneous anisotropic heat-conducting elastic materials. Sharma [8], Sharma and Sharma [9] presented an exact analysis of the free vibrations of simply supported, homogeneous, transversely isotropic cylindrical panel based on the three-dimensional generalized thermoelasticity.

The free vibrations of solid and hollow spheres have been the subject of study for a long period, frequently associated with interest in the oscillations of the earth. In the late nineteenth century, Lamb [10] showed that two basic types of free vibrations namely, 1) the vibrations with zero volume change and zero radial displacement;

and 2) the vibrations with zero radial components of the curl of the displacement, exist in an isotropic sphere. These vibrations are referred as "vibrations of the first and second classes" respectively. Lapwood and Usami [11] named the first class of vibrations as "torsional or toroidal" and the second class as "spheroidal or poloidal". Lapwood and Usami [11] presented an excellent treatment of the vibration of a hollow sphere surrounding a liquid core having finite normal and shear rigidity which serves as an approximate model of the Earth. Lamb [10] derived the equations governing the free vibration of a solid sphere and subsequently Chree [12] obtained the secular equations of free vibrations of a sphere in the more convenient form. Much later, Sâto and Usami [13, 14] computed and tabulated the natural frequency parameters for an extensive set of modes of vibration for the solid sphere. They provided equations and a comprehensive set of results for the distribution of displacement within the vibrating sphere. Shah *et al.* [15,16] studied the vibrations of hollow spheres by using two-dimensional theory of elasticity to obtain natural frequency parameters. Gupta and Singh [17] investigated the problems of wave propagation in transradially isotropic elastic sphere. They showed that, for a transradially isotropic sphere, the toroidal and spheroidal modes of vibrations are independent of each other.

Bargi and Eslami [18] used Green-Lindsay theory of thermoelasticity to study the thermo-elastic response of functionally graded hollow sphere and investigated the material distribution effects on temperature, displacement and stresses. Sharma and Sharma [19], studied the generalized transradially thermoelastic solid sphere.

We have not come across any systematic and exact study on the effect of temperature variations on three dimensional vibration of heat conducting elastic generalized hollow spherical structures. Therefore, the purpose of this paper is to present the exact three dimensional vibration analysis of transradially isotropic, thermoelastic generalized hollow sphere subjected to stress free (or rigidly fixed), thermally insulated (or isothermal) boundary conditions. The secular equations governing three dimensional vibrations in a generalized hollow sphere have been derived by using Frobenius series method. The derived secular equations for spheroidal (S) modes of vibrations which are dependent on thermal variation, have been solved numerically for zinc and cobalt materials in order to compute lowest frequency and dissipation factor. The obtained results in case of toroidal vibrations are found to be in agreement with those of Cohen *et al.* [20].

2. Formulation and Solution

We consider the thermoelastic problem for homogeneous, transradially thermally conducting, elastic generalized hollow sphere of inner and outer radii R_1 and R_2, respectively, initially maintained at uniform temperature T_0 in the undisturbed state. For generalized spherically isotropic thermoelastic medium, in the spherical polar coordinates (r,θ,ϕ,t), the basic governing equations of motion, heat conduction and constitutive relations can be expressed as follows Sharma and Sharma [19].

$$\begin{aligned}&\sigma_{rr,r}+\frac{1}{\mathrm{r}\sin\theta}\sigma_{\mathrm{r}\phi,\phi}+\frac{1}{\mathrm{r}}\sigma_{\mathrm{r}\theta,\theta}\\&\quad+\frac{1}{\mathrm{r}}\left[2\sigma_{rr}-\sigma_{\theta\theta}-\sigma_{\phi\phi}+\sigma_{r\theta}\cot\theta\right]=\rho\,\ddot{\mathrm{u}}_{\mathrm{r}}\end{aligned}\tag{1}$$

$$\begin{aligned}&\sigma_{r\phi,r}+\frac{1}{\mathrm{r}\sin\theta}\sigma_{\phi\phi,\phi}+\frac{1}{\mathrm{r}}\sigma_{\phi\theta,\theta}\\&\quad+\frac{1}{\mathrm{r}}\left[3\sigma_{r\phi}+2\sigma_{\phi\theta}\cot\theta\right]=\rho\,\ddot{\mathrm{u}}_{\phi}\end{aligned}\tag{2}$$

$$\begin{aligned}&\sigma_{r\theta,r}+\frac{1}{\mathrm{r}\sin\theta}\sigma_{\phi\theta,\phi}+\frac{1}{\mathrm{r}}\sigma_{\theta\theta,\theta}\\&\quad+\frac{1}{\mathrm{r}}\left[3\sigma_{r\theta}+\left(\sigma_{\theta\theta}-\sigma_{\phi\phi}\right)\cot\theta\right]=\rho\,\ddot{\mathrm{u}}_{\theta}\end{aligned}\tag{3}$$

$$\begin{aligned}&K_3\left(T_{,rr}+\frac{2}{r}T_{,r}\right)+K_1\left[\frac{1}{r^2}T_{,\theta\theta}+\frac{\cot\theta}{r^2}T_{,\theta}+\frac{1}{r^2\sin^2\theta}T_{,\phi\phi}\right]\\&-\rho C_e\left(\dot{T}+t_0\ddot{T}\right)\\&=T_0\left(\frac{\partial}{\partial t}+t_0\delta_{1l}\frac{\partial^2}{\partial t^2}\right)\left[\beta_1\left(e_{\theta\theta}+e_{\phi\phi}\right)+\beta_3 e_{rr}\right]\end{aligned}\tag{4}$$

where

$$\begin{aligned}&\sigma_{\theta\theta}=c_{11}e_{\theta\theta}+c_{12}e_{\phi\phi}+c_{13}e_{rr}-\beta_1\left(T+t_1\delta_{2l}\dot{T}\right)\\&\sigma_{r\theta}=2c_{44}e_{r\theta}\\&\sigma_{\phi\phi}=c_{12}e_{\theta\theta}+c_{11}e_{\phi\phi}+c_{13}e_{rr}-\beta_1\left(T+t_1\delta_{2l}\dot{T}\right)\\&\sigma_{r\phi}=2c_{44}e_{r\phi}\end{aligned}\tag{5}$$

$$\begin{aligned}&\sigma_{rr}=c_{13}e_{\theta\theta}+c_{13}e_{\phi\phi}+c_{33}e_{rr}-\beta_3\left(T+t_1\delta_{2l}\dot{T}\right)\\&\sigma_{\theta\phi}=2c_{66}e_{\theta\phi},\ e_{rr}=\frac{\partial u_r}{\partial r},\ e_{\theta\theta}=\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\\&e_{\phi\phi}=\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\varphi}+\frac{u_r}{r}+u_\theta\frac{\cot\theta}{r}\\&e_{r\phi}=\frac{1}{2}\left[\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{u_\phi}{r}+\frac{\partial u_\phi}{\partial r}\right]\\&e_{r\theta}=\frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}+\frac{\partial u_\theta}{\partial r}\right]\\&e_{\phi\theta}=\frac{1}{2}\left[\frac{1}{r}\frac{\partial u_\phi}{\partial\theta}-\frac{u_\phi\cot\theta}{r}+\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}\right]\end{aligned}\tag{6}$$

$$\begin{aligned}&\beta_1=\left(c_{11}+c_{12}\right)\alpha_1+c_{13}\alpha_3\\&\beta_3=2c_{13}\alpha_1+c_{33}\alpha_3,\ c_{66}=\left(c_{11}-c_{12}\right)/2\end{aligned}\tag{7}$$

where σ_{rr} is stress along radial direction and $\sigma_{r\theta}$, $\sigma_{r\phi}$

are along tangential direction. Here $\boldsymbol{u}=\left(u_r,u_\theta,u_\phi\right)$ is the displacement vector, $T\left(r,\theta,\phi,t\right)$ is the temperature change, c_{11}, c_{12}, c_{13}, c_{33} and c_{44} are five independent isothermal elasticity; α_1, α_3 and K_1, K_3 are respectively, the coefficients of linear thermal expansion and thermal conductivities along and perpendicular to the axis of symmetry, ρ and C_e are the mass density and specific heat at constant strain and t_0 and t_1 are thermal relaxation times respectively δ_{1l} is Kronecker's delta in which $l=1$ for Lord-Shulman (LS) theory and $l=2$ for Green-Lindsay (GL) theory of thermoelasticity. The comma notation is used for spatial derivatives and the superposed dot denotes time differentiation. It can be proved thermodynamically Sharma and Sharma [9] that $K_1>0, K_3>0$ and of course $\rho>0$ and $T_0>0$. We assume in addition that $C_e>0$ and the isothermal elasticity are components of a positive definite fourth-order tensor. The necessary and sufficient conditions for the satisfaction of latter requirement are

$$c_{11}>0,\ c_{11}>c_{12},\ c_{11}^2>c_{12}^2,\qquad c_{44}>0,\ c_{33}\left(c_{11}+c_{12}\right)>c_{13}^2 \tag{8}$$

3. Boundary Conditions

We consider the free vibrations of a generalized hollow sphere subjected to stress free (or rigidly fixed), thermally insulated (or isothermal) boundary conditions at the surfaces $r=R_1$ (inner radius) and $r=R_2$ (outer radius). Mathematically, this leads to

1) For stress free, thermally insulated (or isothermal) boundary of the sphere.

$$\sigma_{rr}=0,\ \sigma_{r\theta}=0,\ \sigma_{r\phi}=0,\quad T_{,r}=0\ \ (\text{or}\ \ T=0) \tag{9a}$$

at $r=R_1$ (inner radius) and $r=R_2$ (outer radius);

2) For rigidly fixed, thermally insulated (or isothermal) boundary of the sphere.

$$u_r=0,\ u_\theta=0,\ u_\phi=0,\quad T_{,r}=0\ \ (\text{or}\ \ T=0) \tag{9b}$$

at $r=R_1$ (inner radius) and $r=R_2$ (outer radius).

4. Solution of the Problem

We define the dimensionless quantities

$$r'=r/R,\ u_i'=u_i/R,\ \sigma_{ij}'=\frac{\sigma_{ij}}{c_{44}},\varepsilon_T=\frac{\beta_3^2T_0}{\rho C_e c_{44}},\ R_1'=R_1/R$$
$$R_2'=R_2/R,\ c_1=\frac{c_{11}}{c_{44}},c_2=\frac{c_{12}}{c_{44}},\ c_3=\frac{c_{13}}{c_{44}},$$
$$c_4=\frac{c_{33}}{c_{44}},\ c_{66}=\frac{c_{11}-c_{12}}{2},$$
$$\bar{\beta}=\frac{\beta_1}{\beta_3},\ \bar{K}=\frac{K_1}{K_3},\ t'=\frac{v_s}{R}t,\ T'=\frac{T}{T_0},c'=\frac{c}{v_s},\ \Omega^*=\frac{\omega^*R}{v_s},$$
$$\beta^*=\frac{\beta_1T_0}{c_{44}},\ \varepsilon^*=\frac{\varepsilon_T\Omega^*}{\beta^*},\ t_0'=\frac{v_s}{R}t_0,\ t_1'=\frac{v_s}{R}t_1 \tag{10}$$

where $v_s^2=c_{44}/\rho$ and $\omega^*=C_e c_{44}/K_3$ are shear wave velocity and characteristic frequency of the generalized hollow sphere respectively. The primes have been suppressed for convenience.

It is advantageous to express the displacements u_θ, u_ϕ and u_r in terms of functions ψ, G, w and T defined by Sharma and Sharma [19] as.

$$u_\theta=-\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\phi}-\frac{\partial G}{\partial\theta},\ u_\phi=\frac{\partial\psi}{\partial\theta}-\frac{1}{\sin\theta}\frac{\partial G}{\partial\phi},\ u_r=w \tag{11}$$

Using Equation (11) in Equations (1)-(4), we find that

$$\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}-\frac{2}{r^2}+\frac{c_1-c_2}{r^2}+\frac{c_1-c_2}{2r^2}\nabla^2+\frac{\partial^2}{\partial t^2}\right)\psi=0 \tag{12}$$

$$\begin{aligned}&\left(\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{\bar{K}}{r^2}\nabla^2-\Omega^*\left(\frac{\partial}{\partial t}+t_0\frac{\partial^2}{\partial t^2}\right)\right)T\right.\\&\left.-\varepsilon^*\Omega^*\left(\frac{\partial}{\partial t}+t_0\delta_{1l}\frac{\partial^2}{\partial t^2}\right)\left[\left(\frac{\partial}{\partial r}+\frac{2\bar{\beta}}{r}\right)w-\frac{\bar{\beta}}{r}\nabla^2G\right]\right)=0\end{aligned} \tag{13}$$

$$\begin{aligned}&-\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{c_1}{r^2}\nabla^2-\frac{2-c_1+c_2}{r^2}+\frac{\partial^2}{\partial t^2}\right)G\\&+\left(\frac{1+c_3}{r}\frac{\partial}{\partial r}+\frac{2+c_1+c_2}{r^2}\right)w-\frac{\bar{\beta}\beta^*\left(T+t_1\delta_{2l}\dot{T}\right)}{r}=0\end{aligned} \tag{14}$$

$$\begin{aligned}&\left[c_4\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)-\frac{2\left(c_1-c_3+c_2\right)}{r^2}+\frac{1}{r^2}\nabla^2+\frac{\partial^2}{\partial t^2}\right]w\\&-\left[\frac{1+c_3}{r}\left(\frac{\partial}{\partial r}-\frac{1}{r}\right)+\frac{1}{r^2}\left(2c_3-c_1-c_2\right)\right]\nabla^2G\\&-\beta^*\left(\frac{\partial}{\partial r}+\frac{2}{r}-\frac{2\bar{\beta}}{r}\right)\left(T+t_1\delta_{2l}\dot{T}\right)=0\end{aligned} \tag{15}$$

where $\nabla^2=\dfrac{\partial^2}{\partial\theta^2}+\cot\theta\dfrac{\partial}{\partial\theta}+\dfrac{1}{\sin^2\theta}\dfrac{\partial^2}{\partial\phi^2}$

Assume spherical wave solution of the form

$$\begin{aligned}\psi\left(r,\theta,\phi,t\right)&=\frac{1}{\sqrt{r}}\sum_{n=1}^{\infty}U_n\left(r\right)P_n^m\left(\cos\theta\right)e^{i\left(\omega t+m\phi\right)}\\w\left(r,\theta,\phi,t\right)&=\frac{1}{\sqrt{r}}\sum_{n=0}^{\infty}W_n\left(r\right)P_n^m\left(\cos\theta\right)e^{i\left(\omega t+m\phi\right)}\\G\left(r,\theta,\phi,t\right)&=\frac{1}{\sqrt{r}}\sum_{n=1}^{\infty}V_n\left(r\right)P_n^m\left(\cos\theta\right)e^{i\left(\omega t+m\phi\right)}\\T\left(r,\theta,\phi,t\right)&=\frac{1}{\sqrt{r}}\sum_{n=0}^{\infty}T_n\left(r\right)P_n^m\left(\cos\theta\right)e^{i\left(\omega t+m\phi\right)}\end{aligned} \tag{16}$$

where $P_n^m\left(\cos\theta\right)$ is the Legendre polynomial; *n* and *m* are integers and $\omega'=\dfrac{R}{v_s}\omega$ is the dimensionless frequency.

Upon substituting solution (16) in Equations (12)-(15), we obtain

$$\left[\nabla_2^2+1-\frac{\eta^2}{\xi^2}\right]U_n=0 \tag{17}$$

$$\left(\nabla_2^2+\tau_0\Omega^*-\frac{a_4^2}{\xi^2}\right)T_n$$
$$+\varepsilon^*\Omega^*\tau_0'\left[\left(\frac{\partial}{\partial\xi}+\frac{4\bar{\beta}-1}{2\xi}\right)W_n+\frac{\bar{\beta}\left(n^2+n\right)}{\xi}V_n\right]=0 \tag{18.1}$$

$$-\left(\nabla_2{}^2+1-\frac{a_2^2}{\xi^2}\right)V_n+\left[\left(1+c_3\right)\frac{1}{\xi}\frac{\partial}{\partial\xi}+\frac{a_1^2}{\xi^2}\right]W_n$$
$$-\frac{\bar{\beta}\beta^*\tau_1}{\xi}T_n=0 \tag{18.2}$$

$$\left(c_4\nabla_2^2+1-\frac{a_3{}^2}{\xi^2}\right)W_n+n\left(n+1\right)\left[\left(1+c_3\right)\frac{1}{\xi}\frac{\partial}{\partial\xi}-\frac{a_1{}^2}{\xi^2}\right]V_n$$
$$-\tau_1\beta^*\left(\frac{\partial}{\partial\xi}+\frac{3-4\bar{\beta}}{2\xi}\right)T_n=0$$
$$\tag{18.3}$$

where

$$\xi=r\omega,\ \tau_0=t_0-i\omega^{-1},$$
$$\tau_1=t_0\delta_{2l}-i\omega^{-1},\ \tau_0'=t_0\delta_{1l}-i\omega^{-1} \tag{19}$$

The quantities $a_i^2\left(i=1,2,3,4\right)$ and η^2 used in Equa tions (17) and (18) are defined as

$$a_1^2=\frac{3+2\left(c_1+c_2\right)-c_3}{2}$$
$$a_2^2=\frac{9+4\left(c_1\left(n^2+n\right)+c_2-c_1\right)}{4}$$
$$a_3^2=\frac{c_4+4\left(n^2+n\right)+8\left(c_1-c_3+c_2\right)}{4} \tag{20}$$
$$a_4^2=\frac{1+4\bar{K}\left(n^2+n\right)}{4}$$
$$\eta^2=\frac{1}{4}\left[9+2\left(n^2+n-2\right)\left(c_1-c_2\right)\right]$$

The uncoupling of equations for the displacement potential U_n from V_n, W_n and T_n indicates the existence of two distinct modes of vibrations. The solution of Equation (11) for corresponds to the Toroidal mode. There is no effect of temperature and generalization on toroidal mode. Toroidal frequencies will be same as obtained by Cohen *et al*. [20].

The solution of the spherical Bessel Equation (17) is given by

$$U_n\left(\xi\right)=\sum_{n=1}^{\infty}\left(B_{n1}J_\eta\left(\xi\right)+B_{n2}Y_\eta\left(\xi\right)\right),n\geq1 \tag{21}$$

where

$$\eta^2=\frac{1}{4}\left[9+2\left(n^2+n-2\right)\left(c_1-c_2\right)\right]>0$$

and J_η and Y_η is Bessel function of first kind and second kind. B_{n1} and B_{n2} are arbitrary constants determined from the boundary conditions.

Generalized Series Method

The system of Equations (18) has been solved with the help of matrix Frobenius method. Clearly the point $r=0$ (*i.e.* $\xi=0$) is a regular singular point of Equations (18) and all the coefficients of the differential Equations (18) are finite, single valued and continuous in the interval $\xi_1\leq\xi\leq\xi_2$ where $\xi_1=R_1\omega$ and $\xi_2=R_2\omega$. The quantities satisfy all the necessary conditions to have series expansions and hence the Frobenius power series method is applicable to solve the coupled system of differential Equations (18). Thus, we have took the solution vector of the type

$$Y_n=\sum_{k=0}^{\infty}Z_k\xi^{s+k} \tag{22}$$

where $Y_n=\left[W_n\quad V_n\quad T_n\right]'$, $Z_k=\left[A_k\quad B_k\quad D_k\right]'$, where s is a constant (real or complex) to be determined and A_k, B_k, D_k are unknown coefficients to be determined. We need solution in the domain $R_1\leq r\leq R_2$, $R_1>0$. The solution (22) is valid in some deleted interval $0<r<R'$, $R'>R_2$ (about the origin) where R' is the radius of convergence.

Upon substituting solution (22) along with its derivatives in Equations (18) and simplifying, we get

$$\sum_{k=0}^{\infty}\left[H_1\left(s+k\right)\xi^{-2}+H_2\left(s+k\right)\xi^{-1}+H\right]\xi^{s+k}Z_k=0 \tag{23}$$

where $H=diag\left(1,-1,\Omega^*\tau_0\right)$,

$$H_1\left(s+k\right)=\left(H_{ij}\left(s+k\right)\right),\ i,j=1,2,3$$
$$H_2\left(s+k\right)=\left(H_{il}'\left(s+k\right)\right),\ i,l=1,2,3 \tag{24}$$

The elements, H_{ij} and H_{ij}', of matrices H_1 and H_2 are defined as

$$H_{11}\left(s+k\right)=\left(c_4\left(s+k\right)^2-a_3{}^2\right),$$
$$H_{12}\left(s+k\right)=n\left(n+1\right)\left[\left(1+c_3\right)\left(s+k\right)-a_1{}^2\right]$$
$$H_{21}\left(s+k\right)=\left[\left(1+c_3\right)\left(s+k\right)+a_1{}^2\right],$$
$$H_{22}\left(s+k\right)=-\left(\left(s+k\right)^2-a_2{}^2\right)$$
$$H_{33}\left(s+k\right)=\left(\left(s+k\right)^2-a_4{}^2\right), \tag{25}$$
$$H_{23}'\left(s+k\right)=\bar{\beta}\beta^*\Omega^{-1}$$
$$H_{13}'\left(s+k\right)=-\tau_1\Omega^*\left(s+k+\frac{3-4\bar{\beta}}{2}\right),$$
$$H_{31}'\left(s+k\right)=\Omega^*\varepsilon^*\tau_0'\left(s+k+\frac{\left(4\bar{\beta}-1\right)}{2}\right)$$
$$H_{32}'=\Omega^*\varepsilon^*\tau_0'\left(n^2+n\right)$$

Equating to zero the coefficients of lowest powers of ξ (*i.e.* $\xi^{s-2}=0$) in Equation (23), we obtain:

$$H_1(s)Z_0=0 \tag{26}$$

where

$$Z_0=\begin{bmatrix}A_1 & B_1 & D_1\end{bmatrix}',\quad H_1(s)=H_{il}(s)\quad i,l=1,2,3 \tag{27}$$

For the existence of non-trivial solution of Equations (26) we must have $|H_1(s)|=0$, which results in the following indicial equations

$$\begin{aligned}&s^4-As^2+C=0\\&s^2-a_4^2=0\end{aligned} \tag{28}$$

where the coefficients A and C are given by

$$A=\frac{a_3^2+c_4a_2^2-n(n+1)(1+c_3)^2}{c_4},$$

$$C=\frac{a_2^2a_3^2-n(n+1)a_1^4}{c_4}$$

The roots of indicial Equations (28) are given as

$$s_1^2=\frac{A+\sqrt{A^2-4C}}{2},\ s_2^2=\frac{A-\sqrt{A^2-4C}}{2},\quad s_3^2=a_4^2 \tag{29}$$

Clearly the roots $s_j\ (j=1,2,3,4,5,6)$ are related through the relation $s_4=-s_1, s_5=-s_2, s_6=-s_3$, in which s_3 is real but the roots s_1 and s_2 may be, in general, complex. In case the parameter, s, is complex, then leading terms in the complex series solution (22) are of the type:

$$\begin{aligned}&\begin{bmatrix}A_0 & B_0 & D_0\end{bmatrix}'\xi^s=Z_0\xi^{s_R+s_I}\\&=Z_0\xi^{s_R}\left[\cos(s_I\log\xi)+i\sin(s_I\log\xi)\right]\end{aligned} \tag{30}$$

In order to obtain two independent real solutions, according to Neuringer [21], it is sufficient to use any one of the complex root in a part and taking its real and imaginary parts. Also, the treatment of complex case is unlike that of the real root case with the advantage that the differential equation is required to be solved only once in the former case rather than twice as in latter one. For the choice of roots of the indicial equations, the system of Equations (27) leads to following eigen vectors:

$$\begin{aligned}&Z_0(s_1)=Z_0(s_4)=\begin{bmatrix}1 & Q_B(s_1) & 0\end{bmatrix}'M_0,\\&Z_0(s_2)=Z_0(s_5)=\begin{bmatrix}1 & Q_B(s_1) & 0\end{bmatrix}'M_0,\\&Z_0(s_3)=Z_0(s_6)=\begin{bmatrix}0 & 0 & 1\end{bmatrix}'M_0\end{aligned} \tag{31}$$

where

$$Q_B(s_j)=\frac{(1+c_3)s_j+a_1^2}{s_j^2-a_2^2}=-\frac{c_4s_j^2-a_3^2}{n(n+1)\left[(1+c_3)s_j-a_1^2\right]}$$

$(j=1,2,3,4,5,6)$

and A_0 is a constant. Thus we have

$$\begin{aligned}&A_0=\begin{bmatrix}1 & 1 & 0\end{bmatrix}M_0,\\&B_0=\begin{bmatrix}Q_B(1) & Q_B(2) & 0\end{bmatrix}M_0,\\&D_0=\begin{bmatrix}0 & 0 & 1\end{bmatrix}M_0\end{aligned} \tag{32}$$

as the corresponding eigen vectors. Again equating to zero the coefficients of next lowest degree term ξ^{s-1} in Equation (23) and noting that the matrix $H_1(s_j+1)$ is nonsingular for each j, we obtain:

$$Z_1=-H_1(s_j+1)^{-1}H_2(s_j)Z_0=D_1^*Z_0 \tag{33}$$

where

$$\begin{aligned}&D_1^*=-H_1(s_j+1)^{-1}H_2(s_j)=[A_{il}],\\&i,l=1,2,3\end{aligned} \tag{34}$$

The matrices $H_1(s_j+1)$ and $H_2(s_j)$ can be written from the Equation (23) by setting $k=1$ and $[A_{il}]$ are defined in the Appendix A1.

Now equating the coefficients of powers of ξ^{s+k} equal to zero, we obtain following recurrence relation:

$$\begin{aligned}&H_1(s+k+2)Z_{k+2}=-H_2(s+k+1)Z_{k+1}-HZ_k\\&k=0,1,2\cdots\end{aligned} \tag{35}$$

where the matrices H_1, H_2 and H are defined in Equation (23). This implies that

$$\begin{aligned}Z_{k+2}=&-\left(H_1(s_j+k+2)\right)^{-1}\\&\cdot\left[H_2(s_j+k+1)Z_{k+1}+HZ_k\right]\end{aligned} \tag{36}$$

Now putting $k=0,1,2,3\cdots$ in Equation (36) successively and simplifying, we get

$$\begin{aligned}&Z_{k+2}\\&=-\left(H_1(s_j+k+2)\right)^{-1}\left[H_2(s_j+k+1)D_{k+1}^*+HD_k^*\right]Z_0\\&=D_{k+2}^*Z_0\end{aligned} \tag{37}$$

It can be easily shown that the matrix D_{k+2}^* has similar form to that of $H_1(s_j+k+2)$ for even values of k and it is alike $H_2(s_j+k+2)$ for odd values of k. Thus we have:

$$\begin{aligned}&Z_{2k+2}=D_{2k+2}^*Z_0,\ Z_{2k+1}=D_{2k+1}^*Z_0,\\&k=0,1,2,3\cdots\end{aligned} \tag{38}$$

where

$$D^*_{2k+2} = -\left(H_1\left(s_j + 2k + 2\right)\right)^{-1}\left[H_2\left(s_j + 2k + 1\right)D^*_{2k+1} + HD^*_{2k}\right] = \left[K_{ij}\right]_{3\times 3},$$
$$D^*_{2k+1} = -\left(H_1\left(s_j + 2k + 1\right)\right)^{-1}\left[H_2\left(s_j + 2k\right)D^*_{2k} + HD^*_{2k-1}\right] = \left[K'_{ij}\right]_{3\times 3} \quad (39)$$

Here the elements $K_{ij}, K'_{ij}\ (i,\ j = 1,\ 2,\ 3)$ are given by Equations (A2)-(A3) as defined in the Appendix. Moreover, it can be shown that

$$D^*_{2k+2} \approx o\left(k^{-2}\right)D^*,\quad D^*_{2k+1} \approx o\left(k^{-1}\right)D^{**} \quad (40)$$

where

$$D^* = \frac{\varepsilon^*\Omega^*\tau_1}{c_4} diag\left[1 \quad 0 \quad 1\right] \text{ and}$$
$$D^{**} \text{ is } 3\times 3 \text{ null matrix} \quad (41)$$

Noting that both the matrices $D^*_{2k+2} \to 0$ and $D^*_{2k+1} \to 0$, as $k \to \infty$ and using the fact that $\lim_{k\to\infty} P_k = P(\{P_k\} \to P$, if each k^2 component sequence converges (Cullen [22]), we can conclude that the series (22) are absolutely and uniformly convergent with infinite radius of convergence. Therefore, the considered series in Equation (22) is analytic and hence can be differentiated term by term. Moreover, the derived series are also analytic functions.

Thus the general solution of the system of Equations (18) has the form

$$\left\{W_n(\xi),\ V_n(\xi),\ T_n(\xi)\right\} = \sum_{j=1}^{6}\sum_{k=o}^{\infty} C_{jk}\left\{A_k\left(s_j\right),\ B_k\left(s_j\right),\ D_k\left(s_j\right)\right\}\left(\xi\right)^{sj+k} \quad (42)$$

where $s_j\ (j = 1, 2, 3, 4, 5, 6)$ are the eigen-values and $\left\{A_k\left(s_j\right),\ B_k\left(s_j\right), D_k\left(s_j\right)\right\}$ are eigenvectors corresponding to the eigen-values s_j and integer k. The quantities C_{jk} are arbitrary constants to be evaluated. Consequently, the potential functions w, G and T are written from Equations (12) by using (42) as under:

$$\left\{w,\ G,\ T\right\}\left(r,\ \theta,\ \phi, t\right) = r^{\frac{-1}{2}} \sum_{n}^{\infty}\sum_{j=1}^{6}\sum_{k=o}^{\infty} C_{njk}\left\{A_k\left(s_j\right),\ B_k\left(s_j\right),\ D_k\left(s_j\right)\right\}\left(\xi\right)^{sj+k} P_n^m\left(\cos\theta\right)e^{i(m\phi+\omega t)}$$
$$\psi\left(r,\ \theta,\ \phi, t\right) = r^{\frac{-1}{2}}\sum_{n=1}^{\infty}(B_{n1}J_\eta\left(\xi\right) + B_{n2}Y_\eta\left(\xi\right))\, P_n^m\left(\cos\theta\right)e^{i(m\phi+\omega t)} \quad (43)$$

The unknowns B_{n1}, B_{n2} and $C_{njk}\left(j = 1, 2, 3, 4, 5, 6\right)$ can be evaluated by using boundary conditions (9) at the inner and outer boundaries of the generalized hollow sphere.

5. Secular Dispersion Relation

For a generalized spherically isotropic, thermally conducting hollow sphere the stress free (or rigidly fixed), thermally insulated (or isothermal) conditions (9) hold.

5.1. Stress-Free Generalized Hollow Sphere

Upon employing stress free and thermal boundary conditions (9a) at the surface $\xi = \xi_1$ and $\xi = \xi_2$ of the sphere and simplifying we have

$$\sum_{n}^{\infty}\left[\left[E^0_{il}\right]\left[X_0\right] + \sum_{k=1}^{\infty}\left[E^0_{il}\right]\left[X_k\right]\right] = 0 \quad (44)$$
$$\left(i,\ l = 1,\ 2,\ 3,\ 4,\ 5, 6,\ 7,\ 8\right)$$

where

$$X_0 = \left[C_{n10} \quad C_{n20} \quad C_{n30} \quad C_{n40} \quad C_{n50} \quad C_{n60} \quad B_{n1} \quad B_{n2}\right] \quad (45)$$

$$X_k = \left[C_{n1k} \quad C_{n2k} \quad C_{n3k} \quad C_{n4k} \quad C_{n5k} \quad C_{n6k} \quad B_{n1} \quad B_{n2}\right] \quad (46)$$

$$E^0_{11} = \left[\left(n(n+1)c_3Q_B\left(s_1\right) + \left(\frac{4c_3 - c_4}{2} + c_4 s_1\right)\right)\right]A_0\left(s_1\right)\left(\xi_1\right)^{s_1} P$$

$$E^0_{31} = \begin{cases}\left(s_1 + k + \dfrac{1}{2}\right)D_0\left(s_1\right)\left(\xi_1\right)^{s_1} P, \text{ for thermally insulated}\\ D_0\left(s_1\right)\left(\xi_1\right)^{s_1} P, \text{ for isothermal}\end{cases} \quad (47)$$

$$E^0_{51} = \left[1 + \left(\frac{3}{2} - s_1\right)Q_B\left(s_1\right)\right]A_0\left(s_1\right)\left(\xi_1\right)^{s_1}\ \bar{P}P',$$

$$E^0_{71} = \left[1 + \left(\frac{3}{2} - s_1\right)Q_B\left(s_1\right)\right]A_0\left(s_1\right)\left(\xi_1\right)^{s_1}\ P/\bar{P}$$

$$E_{57}^{0}=\sum_{n=1}^{\infty}\left(\xi_1 J_{\eta+1}\left(\xi_1\right)-\left(\eta-\frac{3}{2}\right)J_\eta\left(\xi_1\right)\right)P/\bar{P},\quad E_{58}^{0}=\sum_{n=1}^{\infty}\left(\xi_1 Y_{\eta+1}\left(\xi_1\right)-\left(\eta-\frac{3}{2}\right)Y_\eta\left(\xi_1\right)\right)\bar{P}P' \tag{48}$$

$$E_{11}^{k}=\left[\left(\sum_{n=1}^{\infty}n(n+1)c_3 Q_{B1}\left(s_1\right)+\sum_{n=0}^{\infty}\left(\frac{4c_3-c_4}{2}+c_4\left(s_1+k+1\right)\right)A_{k+1}\left(s_1\right)-i\tau_1\beta^{*}D_k\left(s_1\right)\right)\right]\left(\xi_1\right)^{s_1+k}P$$

$$E_{31}^{k}=\begin{cases}\sum_{n=0}^{\infty}\left(s_1+k+\frac{1}{2}\right)D_{k+1}\left(s_1\right)\left(\xi_1\right)^{s_1+k}P, & \text{for therma lly insulated}\\ \sum_{n=0}^{\infty}D_k\left(s_1\right)\left(\xi_1\right)^{s_1+k}P, & \text{for isothermal}\end{cases}$$

$$E_{51}^{k}=\left[\sum_{n=0}^{\infty}1+\sum_{n=1}^{\infty}\left(\frac{1}{2}-s_1-k\right)Q_B\left(s_1\right)\right]A_k\left(\xi_1\right)^{s_1+k}\bar{P}P', \tag{49}$$

$$E_{il}^{0}=E_{il}^{k}\quad for\ \ i=5,6,7,8,\ \text{and}\ \ l=7,8$$

$$E_{71}^{k}=\left[\sum_{n=0}^{\infty}1+\sum_{n=1}^{\infty}\left(\frac{1}{2}-s_1-k\right)Q_B\left(s_1\right)\right]A_k\left(\xi_1\right)^{s_1+k}P/\bar{P}$$

Here the elements $E_{1l}^{k}=E_{1l}^{0}\left(for\ k\geq 0\right)\left(l=2,3,4,5,6\right)$ of determinant Equations (47) and (49) can be obtained by just replacing $s_l,\ l=1$ in $E_{il}\left(i=1,3,5,7\right)$ with $s_l,\ l=2,3,4,5,6$, while $E_{il}\left(i=2,4,6,8\right)$ are obtained by replacing ξ_1 in $E_{il}\left(i=1,3,5,7\right)$ with ξ_2. The element $E_{il},\ i=5,6,7,8$ and $l=8$ in (48) can be obtained by replacing Bessel's function of first kind J_η with that of second kind Y_η and the elements $E_{il}\ i=6,8$ and $l=7,8$ can be obtained by replacing ξ_1 in $E_{il}\ i=5,7$ and $l=7,8$ with ξ_2 respectively.

The Equation (44) holds iff each term vanishes separately. This implies that

$$\left[E_{il}^{0}\right]\left[X_0\right]=0\ \text{ for } k=0,\ n>0 \tag{50}$$

$$\left[E_{il}^{k}\right]\left[X_k\right]=0\ \text{ for } k>0,\ n\geq 0 \tag{51}$$

The Equations (50) and (51) have a non-trivial solution iff

$$\left|E_{il}^{0}\right|=0,\ \left(i,\ l=1,2,3,4,5,6,7,8\right)\ \text{ for }\ k=0, \tag{52}$$

$$\left|E_{il}^{k}\right|=0,\ \left(i,\ l=1,2,3,4,5,6,7,8\right)\ \text{ for }\ k>0,\ n\geq 0 \tag{53}$$

After lengthy but straightforward simplifications and reductions, the determinant Equations (52) and (53) lead to the following secular equations.

$$E_{77}E_{88}-E_{78}E_{87}=0\quad for\quad n\geq 1 \tag{54}$$

$$\left(\left(1-\mu^2\right)\frac{\mathrm{d}P_n\left(\mu\right)}{\mathrm{d}\mu}-mP_n\left(\mu\right)\right)=0\ \ for\, n\geq 1 \tag{55}$$

$$\det\left(E_{ij}^{0}\right)=0\ ,\ i\ ,\ l=1,2,3,4,5,6\ \ k=0,\ n>0 \tag{56}$$

$$\det\left(E_{ij}^{k}\right)=0\ ,\ i\ ,\ l=1,2,3,4,5,6\ \ k>0,\ n\geq 0 \tag{57}$$

where

$$E_{11}^{0}=\left[\left(n(n+1)c_3+\left(\frac{4c_3-c_4}{2}+c_4\left(s_1\right)\right)\right)\right]A_0\left(s_1\right)\left(\xi_1\right)^{s_1}$$

$$E_{31}^{0}=\begin{cases}\left(s_1+\frac{1}{2}\right)D_0\left(s_1\right)\left(\xi_1\right)^{s_1}, & \text{for thermally insulated}\\ D_0\left(s_1\right)\left(\xi_1\right)^{s_1}, & \text{for isothermal}\end{cases} \tag{58}$$

$$E_{51}^{0}=\left[1+\left(\frac{1}{2}-s_1-k\right)Q_B\left(s_1\right)\right]A_0\left(s_1\right)\left(\xi_1\right)^{s_1}$$

$$E_{77}=\left(\xi_1 J_{\eta+1}\left(\xi_1\right)-\left(\eta-\frac{3}{2}\right)J_\eta\left(\xi_1\right)\right)$$

$$E_{11}^{k}=\left[\left(n(n+1)c_3 B_{k+1}\left(s_1\right)+\left(\frac{4c_3-c_4}{2}+c_4\left(s_1+k+1\right)\right)A_{k+1}\left(s_1\right)-i\tau_1\beta^{*}D_k\left(s_1\right)\right)\right]\left(\xi_1\right)^{s_1+k}$$

$$E_{31}^{k}=\begin{cases}\left(s_1+k+\frac{1}{2}\right)D_{k+1}\left(s_1\right)\left(\xi_1\right)^{s_1+k}, & \text{for thermally insulated}\\ D_k\left(s_1\right)\left(\xi_1\right)^{s_1+k}, & \text{for isothermal}\end{cases} \tag{59}$$

$$E_{51}^{k}=\left[Q_B\left(s_1\right)+\left(\frac{1}{2}-s_1-k\right)\right]A_{k+1}\left(s_1\right)\left(\xi_1\right)^{s_1+k}$$

The elements $E_{il}^{0}\left(j=2,3,4,5,6\right)$ of determinant Equations (58) and (59) can be obtained by just replacing $s_l, l=1$ in $E_{il}^{0}\left(i=1,3,5\right)$ with $s_l,\ l=2,3,4,5,6$ while $E_{il}^{0}\left(i=2,4,6\right)$ are obtained by replacing ξ_1 in $E_{il}^{0}\left(i=1,3,5\right)$ with ξ_2. The element E_{78} can be obtained by replacing Bessel's function of first kind J_η with that of second kind Y_η and the elements E_{87} can be obtained by replacing ξ_1 in E_{87} and E_{88} with ξ_2 respectively.

The secular dispersion Equation (54) provides us first class vibrations called Toroidal vibrations (T-modes) as discussed by Cohen *et al*. [20]. Clearly these modes do not depend on thermal variations as expected. These are characterized by the absence of radial component of displacement.

5.1.1. Spheroidal Mode

The secular Equations (56) and (57) govern the second class vibrations called Spheroidal vibrations (S-modes) for $n>0$, $k=0$ and $k>0, n=0$ and $n>0$ respectively. These relations contain complete information regarding frequency and other characteristics of the Spheroidal modes of vibrations in a transradially generalized isotropic hollow sphere. The detail of Spheroidal vibrations on neglecting the thermal effects and considering only the radial vibrations have been discussed by Ding *et al*. [23] in case of elastokinetics.

5.1.2. Toroidal Mode

The secular dispersion Equation (54) provides us

$$\tan\left(t^*\omega\right)=\frac{3t^*\omega}{3-\left(t^*\omega\right)^2},\ \text{for}\ \ n=1 \tag{60}$$

$$\left(\eta-1\right)J_{\eta+1/2}\left(t^*\omega\right)-\left(t^*\omega\right)J_{\eta+3/2}\left(t^*\omega\right)=0,\ \text{for}\ \ n>1 \tag{61}$$

where t*(=h/R) is thickness to mean radius ratio, where thickness of the sphere is defined as $h=R_2-R_1$ and mean radius as $R=\dfrac{R_2+R_1}{2}$.

Clearly these modes do not depend on thermal variations as expected. The Equations (60) agree with Ding *et al*. [23] and are characterized by the absence of radial component of displacement. For homogeneous isotropic solid we have

$$\begin{aligned}&c_{11}=c_{33}=\lambda+2\mu,\ c_{12}=c_{13}=\lambda,\ c_{44}=\mu,\\&\beta_1=\beta=\beta_3,\ K_1=K=K_3.\end{aligned} \tag{62}$$

so that $\eta=n+\dfrac{1}{2}$ and the secular Equation (61) reduces to

$$\left(n-1\right)J_{n+1/2}\left(t^*\omega\right)-\left(t^*\omega\right)J_{n+3/2}\left(t^*\omega\right)=0\quad \text{for}\ \ n>1 \tag{63}$$

It can be shown that Equation (63) is identical to the one obtained by Love [24], [page 284, Equation (38)]. These modes have also been discussed in detail by Cohen *et al*. [20] and Ding *et al*. [23] and the corresponding frequencies of such Toroidal modes are same as in case of elastokinetics. The analysis in case of coupled thermoelasticity (CT) can be obtained by setting $t_0=0=t_1$ and for uncoupled thermoelasticity (UCT) by taking $\varepsilon=0$, $t_0=0=t_1$ in the present study. The secular equations and all other relevant results in case of LS and GL theories of dynamic generalized thermoelasticity can be obtained from the above analysis by taking $l=1$ and $l=2$, respectively, in Equations (4) and (5) then using the resulting values of these parameters in different relations at various stages.

5.2. Rigidly Fixed Sphere

Taking the rigidly fixed and thermal boundary conditions (9b) at the surface $\xi=\xi_1$ and $\xi=\xi_2$ of the sphere on displacements we get following determinantal equations

$$\left|d_{ij}\right|=0\quad i,j=1,2,3,4,5,6 \tag{64}$$

$$J_\eta\left(\xi_1\right)Y_\eta\left(\xi_2\right)-J_\eta\left(\xi_2\right)Y_\eta\left(\xi_1\right)=0 \tag{65}$$

$$\left(1-\mu^2\right)\frac{\mathrm{d}P_n\left(\mu\right)}{\mathrm{d}\mu}-mP_n\left(\mu\right)=0 \tag{66}$$

where $\mu=\cos\theta$,

$$\begin{aligned}&d_{11}=A_k\left(s_1\right)\left(\xi_1\right)^{s_1+k}\\&d_{31}=B_k\left(s_1\right)\left(\xi_1\right)^{s_1+k}\\&d_{51}=\begin{cases}\left(s_1-\dfrac{1}{2}\right)D_k\left(s_1\right)\left(\xi_1\right)^{s_1+k}, & \text{for thermally insulated}\\ D_k\left(s_1\right)\left(\xi_1\right)^{s_1+k}, & \text{for isothermal}\end{cases}\end{aligned} \tag{67}$$

The elements $d_{il}\left(l=2,3,4,5,6\right)$ of determinantal Equation (64) can be obtained by just replacing $s_l, l=1$ in $d_{il}\left(l=1,3,5\right)$ with $s_{il}, l=2,3,4,5,6$, while $d_{il's}\left(i=2,4,6\right)$ are obtained by replacing ξ_1 in $d_{il}\left(i=1,3,5\right)$ with ξ_2.

5.2.1. Toroidal Vibrations

Equation (65) corresponds to first class vibrations (Toroidal mode). Clearly these modes do not depend on thermal and relaxation time variations as expected.

5.2.2. Spheroidal Vibrations

The secular Equations (64) govern the Spheroidal vibrations (S-modes) in case of $n\geq 0, k=0$ and $n\geq 0, k\geq 1$, respectively in a transradial isotropic Generalized thermoelastic hollow sphere subjected to stress free boundary conditions. These relations contain complete information regarding frequency and other characteristics of the Spheroidal modes of vibrations in a transradially isotropic Generalized thermoelastic hollow sphere.

6. Numerical Results and Discussion

We consider the case of free vibrations of a transradially, isotropic generalized thermoelastic hollow spheres made up of solid helium and magnesium materials whose physical data is given in **Table 1**. As given by Sharma and Sharma [19] and Dhaliwal and Singh [26].

Due to the presence of dissipation term in heat conduction Equation (4), the secular equations are in general complex transcendental equations which provide us complex values of the frequency (ω). The real part of frequency (ω) gives us lowest frequency ($\Omega = \omega_R R\left(\rho/c_{44}\right)^{1/2}$) and imaginary part provide us dissipation factor ($D = \omega_I R\left(\rho/c_{44}\right)^{1/2}$), where $\omega_R = \mathrm{Re}\left(\omega\right)$ and $\omega_I = I_m\left(\omega\right)$, for fixed values of n and k. The computer simulated profiles of lowest frequency $\left(\Omega\right)$ and dissipation factor $\left(D\right)$ in spheres of solid helium and magnesium materials have been presented in **Figures 1** to **12** for different values of thickness to mean radius ratio (t*) in the context of LS, GL, and CT theories of coupled thermoelasticity. The values of the thermal relaxation time has been estimated from Equation (2.5) of Chandrasekharaiah [27] $v_{iz}.t_0 = \dfrac{3K}{\rho C_e c_{44}^2}$ and t_1 has been taken proportional to that of t_0. The same numerical technique has been used as given by Sharma and Sharma [19].

Figures 1 and **3** show the variations of lowest frequency (Ω) with thickness to mean radius ratio (t*) for different values of degree of spherical harmonics (n) for solid helium and magnesium respectively in case of stress free generalized hollow sphere. From **Figure 3**, it is observed that the profile of lowest frequency increases parabolically with the increase of t* and the order with respect to generalized theories of thermoelasticity is Ω(CT) > Ω(GL) >Ω(LS) for n = 1 and n = 2. **Figure 3** reveals that the lowest frequency varies linearly and remains dispersionless at all values of the degree of spherical harmonics (n) in the context of LS, GL and CT, generalized theories of thermoelasticity. It has been observed that the order for lowest frequency interlaces is Ω(LS) > Ω(CT) > Ω(GL) for n = 1 and n = 2. It can be concluded that with the increase of degree of spherical harmonics (n), lowest frequency increases. From both the Figures we conclude that with the increase of t* lowest frequency increases.

Figures 2 and **4** show the variations of damping factor (D) with thickness to mean radius ratio (t*) for different values of degree of spherical harmonics (n) for solid he-

Table 1. Physical data for helium and magnesium crystals.

Quantity	Units	Helium	Magnesium
c_{11}	Nm^{-2}	0.4040×10^{10}	5.974×10^{10}
c_{12}	Nm^{-2}	0.2120×10^{10}	2.624×10^{10}
c_{13}	Nm^{-2}	0.01050×10^{10}	2.17×10^{10}
c_{33}	Nm^{-2}	0.5530×10^{10}	6.17×10^{10}
c_{44}	Nm^{-2}	0.1245×10^{10}	3.278×10^{10}
β_1	$\mathrm{Nm}^{-2}\cdot\mathrm{deg}^{-1}$	2.3620×10^{6}	2.68×10^{6}
β_3	$\mathrm{Nm}^{-2}\cdot\mathrm{deg}^{-1}$	2.641×10^{6}	2.68×10^{6}
K_1	$\mathrm{Wm}^{-1}\cdot\mathrm{deg}^{-1}$	0.3×10^{2}	1.7×10^{2}
K_3	$\mathrm{Wm}^{-1}\cdot\mathrm{deg}^{-1}$	0.2×10^{1}	1.7×10^{2}
ε		0.04162	0.0202
ω^*	s^{-1}	1.9890×10^{13}	3.58×10^{11}
t_0	s	0.0000091×10^{-13}	0.279×10^{-11}

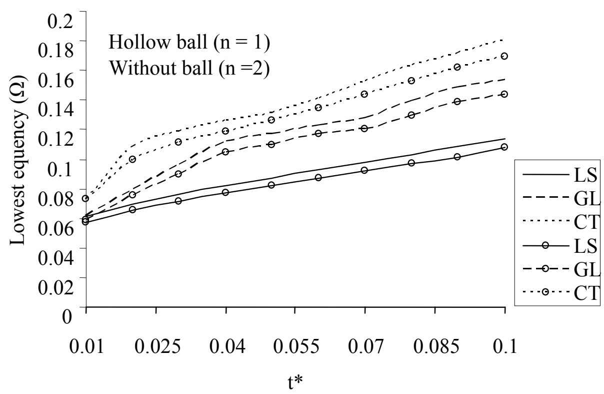

Figure 1. Variation of lowest frequency with thickness to mean radius ratio for different values of degree of spherical harmonics (n) for helium material in case of stress free boundary condition.

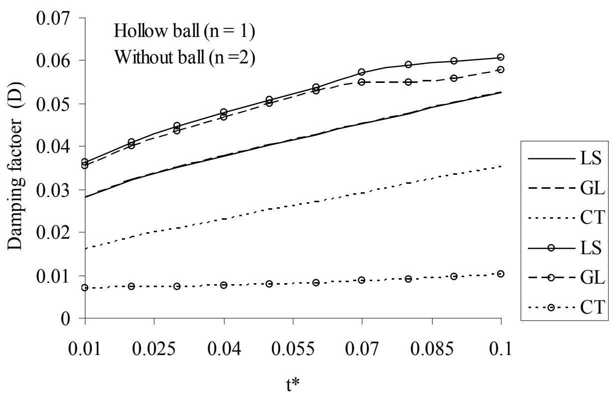

Figure 2. Variation of damping factor with thickness to mean radius ratio for different values of degree of spherical harmonics (n) for helium material in case of stress free boundary condition.

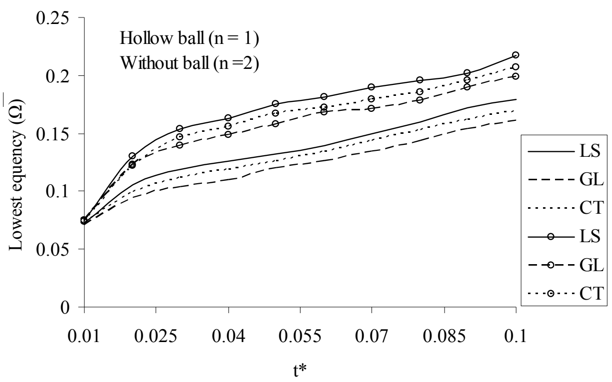

Figure 3. Variation of lowest frequency with thickness to mean radius ratio for different values of degree of spherical harmonics (n) for magnesium material in case of stress free boundary condition.

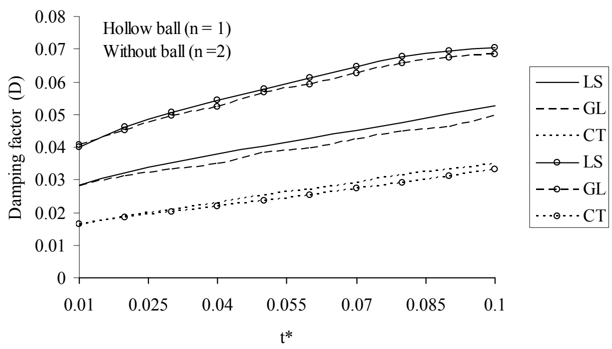

Figure 4. Variation of damping factor with thickness to mean radius ratio for different values of degree of spherical harmonics (n) for magnesium material in case of stress free boundary condition.

lium and magnesium crystals respectively in case of stress free boundary condition. From **Figures 2** and **4** it is observed that the trend of profile of damping increases linearly with t* and the order is $D(LS)>D(GL)\geq D(CT)$ for n = 1 and $D(LS)>D(GL)\geq D(CT)$ for n = 2 respectively.

An examination is made on the variations of dimensionless lowest frequency with respect to thickness to mean radius ratio (t*) ranging from thin spherical shell (t* = 0.05) to the thick spherical shell (t* = 0.1) of isotropic materials. The findings confirm that the variation of the Spheroidal frequencies increases with t* and for degree of spherical harmonics (n), the same trend of profile for damping has been observed *i.e.* damping increases with the increase of t*.

Figures 5 and **7** show the variations of lowest frequency (Ω) with thickness to mean radius ratio (t*) for different values of degree of spherical harmonics (n) for two materials solid helium and magnesium respectively in case of rigidly fixed boundary condition. For both the materials similar behaviour has been observed *i.e.* the lowest frequency increases with the increase of t*. In

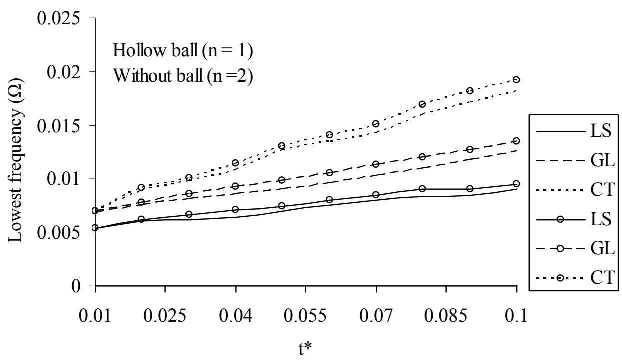

Figure 5. Variation of lowest frequency with thickness to mean radius ratio for different values of degree of spherical harmonics (n) for helium material in case of rigidly fixed boundary condition.

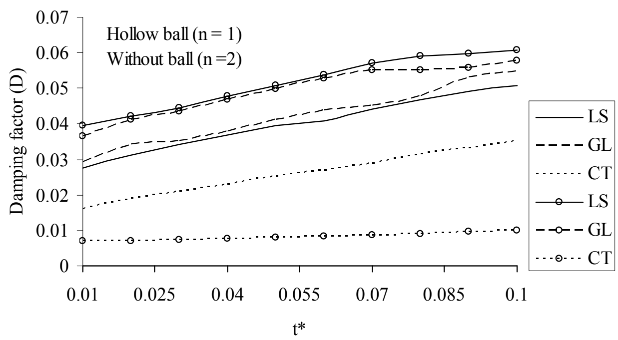

Figure 6. Variation of damping factor with thickness to mean radius ratio for different values of degree of spherical harmonics (n) for helium material in case of rigidly fixed boundary condition.

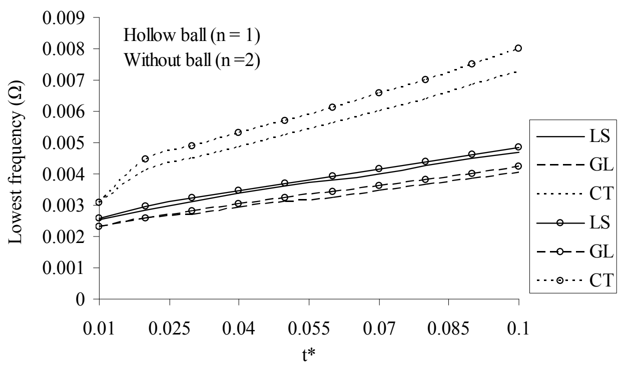

Figure 7. Variation of lowest frequency with thickness to mean radius ratio for different values of degree of spherical harmonics (n) for magnesium material in case of rigidly fixed boundary condition.

Figure 5 the trend of profiles of lowest frequency interlaces is $\Omega(CT)>\Omega(GL)>\Omega(CT)$ for n = 1 and n = 2. In **Figure 7** the profile of lowest frequency interlaces is $\Omega(CT)>\Omega(LS)\geq\Omega(GL)$ in context of linear theories of generalized thermoelasticity.

Figures 6 and **8** show the variations of damping factor (D) with degree of spherical harmonics (n) for two mate-

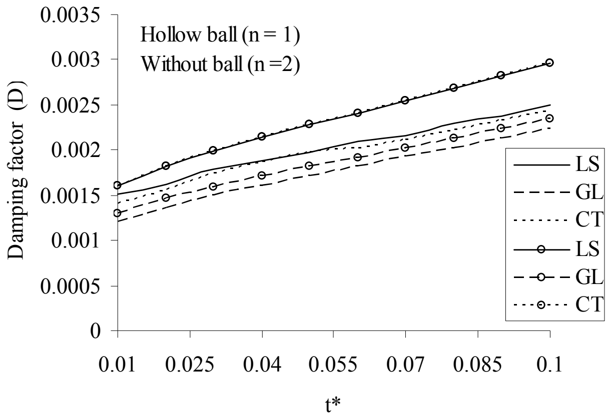

Figure 8. Variation of damping factor with thickness to mean radius ratio for different values of degree of spherical harmonics (n) for magnesium material in case of rigidly fixed boundary condition.

rials solid helium and magnesium respectively, for n = 1 and 2. From both the Figures it is observed that the trend of profile of damping factor increases linearly with t*. In **Figure 6** the order of damping is D(LS) > D(GL) > D(CT) for n = 2 and the profile of damping interlaces as D(GL) > D(LS) > D(CT) for n = 1. In **Figure 8** the order of damping is D(LS) = D(CT) > D(GL) for n = 2 and D(LS) > D(CT) > D(GL) for n = 1.

Figures 9 and **11** show the variations of lowest frequency (Ω) with time ratio for different values of thickness to mean radius ratio (t*) for solid helium and magnesium respectively. The profiles of lowest frequency are observed to be significantly affected with thermal relaxation time in both the materials with increasing values of thickness to mean radius ratio (t*). From both the Figures it is observed that with the increase of time ratio lowest frequency is nearly constant value, also lowest frequency has more value for thick shell as compared to thin shells. **Figures 10** and **12** show the variations of damping factor (*D*) with time ratio for different values thickness to mean radius ratio (t*) for solid helium and magnesium respectively. **Figure 10** shows that the magnitude of dissipation is constant in the region $0 \le t_1/t_0 \le 0.2$ and there is sharp decrease in the region $0.2 < t_1/t_0 < 0.4$ and then for $t_1/t_0 > 0.4$ there is slow decrease in damping. In **Figure** 12 there is sharp decrease in damping up to $t_1/t_0 = 0.6$ and for $t_1/t_0 > 0.6$ damping increases. From both the figures it is observed that with the increase of time ratio the magnitude of damping factor is nearly constant value, also lowest frequency has more value for thick shells as compared to thin shells.

7. Conclusions

The effect of thermal variations and thermal relaxation time on lowest frequency and dissipation factor of spherical vibrations in generalized thermoelastic hollow sphere has been investigated in the context of LS and GL theories of thermoelasticity with the help of Matrix Fröbenius method. The numerical computations have been done with the help of MATLAB files.

It is noticed that the first class vibrations are not affected by temperature change, thermal relaxation time and remain independent of rest of the motion. The obtained results are similar with the corresponding results

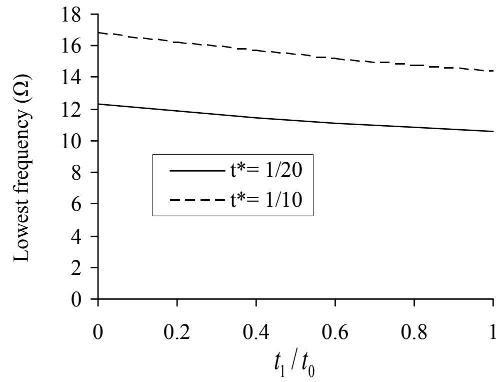

Figure 9. Variation of lowest frequency with time ratio for various value of thickness to mean radius ratio (t*) for helium material.

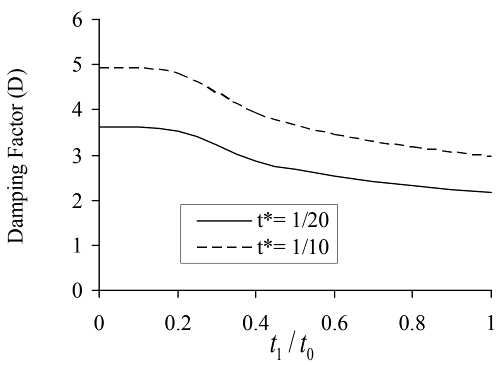

Figure 10. Variation of damping factor with time ratio for various value of t* for helium material.

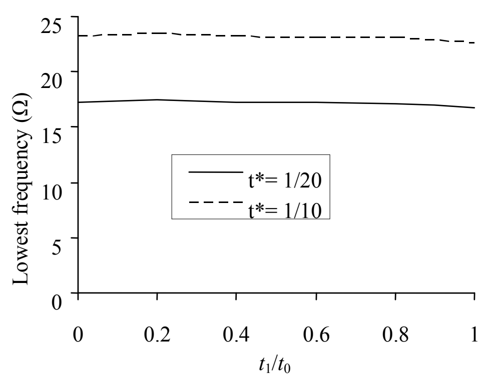

Figure 11. Variation of lowest frequency with time ratio for various value of t* for magnesium material.

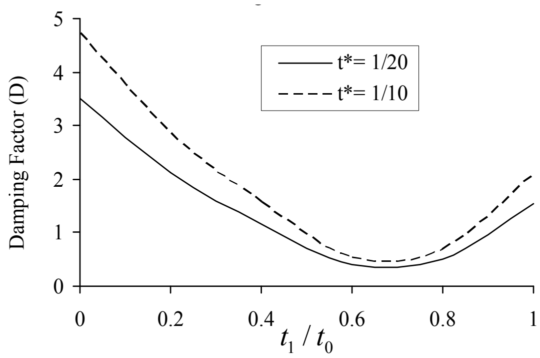

Figure 12. Variation of damping factor with time ratio for various value of t* for magnesium material.

obtained by Love [24], Lamb [10] and Cohen *et al*. [20] in elastokinetics. The lowest frequency of spheroidal vibrations is noticed to be significantly affected due to both temperature variations and relaxation time, in hollow spheres of both helium and magnesium materials.

The lowest frequency and dissipation factor have shown strong dependency on the degree of spherical harmonics (n) and hence the importance of the degree of spherical harmonics must be taken into consideration while designing a spherical structure.

The effect of relaxation time ratio on lowest frequency and dissipation factor of vibration under consideration is also observed in hollow spheres of helium and magnesium materials. This study may find applications in aerospace, navigation, geophysics tribology and other industries where spherical structures are in frequent use.

REFERENCES

[1] W. Nowacki, "Dynamic Problem of Thermoelasticity," Noordhoff, Leyden, 1975.

[2] H. W. Lord and Y. Shulman, "The Generalized Dynamical Theory of Thermoelasticity," *Journal of Mechanics and Physics of Solids*, Vol. 15, No. 5, 1967, pp. 299-309. doi:10.1016/0022-5096(67)90024-5

[3] A. E. Green and K. A. Lindsay, "Thermoelasticity," *Journal of Elasticity*, Vol. 2, No. 1, 1972, pp. 1-7. doi:10.1007/BF00045689

[4] R. S. Dhaliwal and R. S. Sherief, "Generalized Thermoelasticity for Anisotropic Media," *Quarterly of Applied Mathematics*, Vol. 38, No. 3, 1980, pp. 1-8.

[5] D. S. Chandrasekharaiah, "Thermoelasticity with Second Sound—A Review," *Applied Mechanics Review*, Vol. 39, No. 3, 1986, pp. 355-376. doi:10.1115/1.3143705

[6] C. C. Ackerman, B. H. Bentman, A. Fairbank and R. A. Krumhansal, "Second sound in helium," *Physical Review Letters*, Vol. 16, No. 18, 1966, pp. 789-791. doi:10.1103/PhysRevLett.16.789

[7] H. Singh and J. N. Sharma, "Generalized Thermoelastic Waves in Transversely Isotropic," *Journal of Acoustic Society of America*, Vol. 77, No. 3, 1985, pp. 1046-1053. doi:10.1121/1.392391

[8] J. N. Sharma, "Three Dimensional Vibration Analysis of a Homogeneous Transversely Isotropic Cylindrical Panel," *Journal of Acoustical Society of America*, Vol. 110, No. 1, 2001, pp. 254-259. doi:10.1121/1.1378350

[9] J. N. Sharma and P. K. Sharma, "Free Vibration Analysis of Homogeneous Transversely Isotropic Thermoelastic Cylindrical Panel," *Journal of Thermal Stresses*, Vol. 25, No. 2, 2002, pp. 169-182. doi:10.1080/014957302753384405

[10] H. Lamb, "On the Vibrations of an Elastic Sphere," *Proceedings of the London Mathematical Society*, Vol. 13, No. 1, 1882, pp. 189-212. doi:10.1112/plms/s1-13.1.189

[11] E. R. Lapwood and T. Usami, "Free Oscillations of the Earth," Cambridge University Press, Cambridge, 1981.

[12] C. Chree, "The Equations of an Isotropic Elastic Solid in Polar and Cylindrical Co-Ordinates," *Their Solution Transactions of the Cambridge Philosophical Society*, Vol. 14, 1889, pp. 250-269.

[13] Y. Sato and T. Usami, "Basic Study on the Oscillation of Homogeneous Elastic Sphere-Part I," *Frequency of the Free Oscillations Geophysics Magazine*, Vol. 31, No. 1, 1962, pp. 15-24.

[14] Y. Sato and T. Usami, "Basic Study on the Oscillation of Homogeneous Elastic Sphere-Part II Distribution of Displacement," *Geophysics Magazine*, Vol. 31, No. 1, 1962, pp. 25-47.

[15] A. H. Shah, C. V. Ramkrishana and S. K. Datta, "Three Dimensional and Shell Theory Analysis of Elastic Waves in a Hollow Sphere: Part I—Analytical Foundation," *Journal of Applied Mechanics*, Vol. 36, No. 17, 1969, pp. 431-439. doi:10.1115/1.3564698

[16] A. H. Shah, C. V. Ramkrishana and S. K. Datta, "Three Dimensional and Shell Theory Analysis of Elastic Waves in a Hollow Sphere: Part II—Numerical Results," *Journal of Applied Mechanics*, Vol. 36, No. 3, 1969, pp. 440-444. doi:10.1115/1.3564699

[17] A. Gupta and S. J. Singh, "Toroidal Oscillations of a Transradially Isotropic Elastic Sphere," *Proceeding of Indian Academy of Science*, Vol. 99, No. 3, 1990, pp. 383-391.

[18] A. Bargi and M. R. Eslami, "Analysis of Thermoelastic Waves in Functionally Graded Hollow Spheres Based on the Green-Lindsay Theory," *Journal of Thermal Stresses*, Vol. 30, No. 12, 2007, pp. 1175-1193. doi:10.1080/01495730701519508

[19] J. N. Sharma, "Vibration Analysis of Homogeneous Transradially Isotropic Generalized Thermoelastic Spheres", *Journal of Vibration and Acoustics*, Vol. 133, No. 4, 2011, Article ID: 021004. doi:10.1115/1.4003396

[20] H. Cohen, A. H. Shah and C. V. Ramakrishna, "Free Vibrations of a Spherically Isotropic Hollow Sphere," *Acoustica*, Vol. 26, 1972, pp. 329-333.

[21] J. L. Neuringer, "The Frobenius Method for Complex Roots of the Indicial Equation," *International Journal of Mathematics Education Science Technology*, Vol. 9, No. 1, 1978, pp. 71-77. doi:10.1080/0020739780090110

[22] C. G. Cullen, "Matrices and Linear Transformations,"

Addison-Wesley Publishing Company, London, 1966.

[23] H. J. Ding, W. Q. Chen and L. Zhang, "Elasticity of Transversely Isotropic Materials (Series: Solid Mechanics and Its Applications)," Springer, Dordrecht Netherland, 2006.

[24] A. E. |H. Love, "A Treatise on the Mathematical Theory of Elasticity," Cambridge University Press, Cambridge, 1927.

[25] W. Q. Chen and H. J. Ding, "Free Vibration of Multi-Layered Spherically Isotropic Hollow Spheres," *International Journal of Mechanical Sciences*, Vol. 43, No. 3, 2001, pp. 667-680. doi:10.1016/S0020-7403(00)00044-8

[26] R. S. Dhaliwal and A. Singh, "Dynamic Coupled Thermoelasticity," Hindustan, New Delhi, 2001.

[27] D. S. Chandrasekharaiah, "Hyperbolic Thermoelasticity, A Review of Recent Literature," *Applied Mechanics Review*, Vol. 51, No. 12, 1998, pp. 705-729. doi:10.1115/1.3098984

Appendix

The quantities $A_{il}\left(s_j\right),\ i,\ l = 1,2,3$ used in Equations (34) are defined as

$$
\begin{aligned}
&A_{11} = 0\\
&A_{12} = 0\\
&A_{13} = \frac{H_{22}\left(s_j+1\right)H'_{13}\left(s_j\right) - H_{12}\left(s_j+1\right)H'_{23}\left(s_j\right)}{H_{11}\left(s_j+1\right)H_{22}\left(s_j+1\right) + H_{12}\left(s_j+1\right)H_{21}\left(s_j+1\right)}\\
&A_{21} = 0,\quad A_{12} = 0\\
&A_{23} = \frac{-H_{21}\left(s_j+1\right)H'_{13}\left(s_j\right) + H_{11}\left(s_j+1\right)H'_{23}\left(s_j\right)}{H_{11}\left(s_j+1\right)H_{22}\left(s_j+1\right) + H_{12}\left(s_j+1\right)H_{21}\left(s_j+1\right)}\\
&A_{31} = \frac{H'_{31}\left(s_j\right)}{H_{33}\left(s_j+1\right)}\\
&A_{32} = \frac{H'_{31}\left(s_j\right)}{H_{33}\left(s_j+1\right)},\\
&A_{33} = 0
\end{aligned}
\tag{A1}
$$

The quantities $K_{ij}\left(s_j\right),\ K'_{ij}\left(s_j\right)\ \left(i,\ j = 1,\ 2,\ 3\right)$ used in Equations (39) are by

$$
\begin{aligned}
K_{11} = {} & \frac{H_{22}\left(s_j+2k+2\right)}{H^*\left(s_j+2k+2\right)}\left(H'_{13}\left(s_j+2k+1\right)\frac{H'_{31}\left(s_j+2k\right)}{H_{33}\left(s_j+2k+1\right)}+1\right)\\
& -\frac{H_{12}\left(s_j+2k+2\right)}{H^*\left(s_j+2k+2\right)}H'_{23}\left(s_j+2k+2\right)\left(\frac{H_{21}\left(s_j+2k+1\right)}{H^*\left(s_j+2k+1\right)}H'_{13}\left(s_j+2k\right)+\frac{H_{11}\left(s_j+2k+1\right)}{H^*\left(s_j+2k+1\right)}H'_{23}\right)\\
K_{12} = {} & \frac{H_{22}\left(s_j+2k+2\right)}{H^*\left(s_j+2k+2\right)}H'_{13}\left(s_j+2k+1\right)\frac{H'_{32}\left(s_j+2k\right)}{H_{22}\left(s_j+2k+1\right)}-\frac{H_{12}\left(s_j+2k+2\right)}{H^*\left(s_j+2k+2\right)}\left(\frac{H'_{23}\left(s_j+2k+1\right)H'_{32}\left(s_j+2k\right)}{H_{33}\left(s_j+2k+1\right)}-1\right)\\
K_{21} = {} & \frac{-H_{21}\left(s_j+2k+2\right)}{H^*\left(s_j+2k+2\right)}\left(H'_{13}\left(s_j+2k+1\right)\frac{H'_{31}\left(s_j+2k\right)}{H_{33}\left(s_j+2k+1\right)}+1\right)\\
& +\frac{H_{11}\left(s_j+2k+2\right)}{H^*\left(s_j+2k+2\right)}H'_{23}\left(s_j+2k+2\right)\left(\frac{-H_{21}\left(s_j+2k+1\right)}{H^*\left(s_j+2k+1\right)}H'_{23}\left(s_j+2k\right)+\frac{H_{11}\left(s_j+2k+1\right)}{H^*\left(s_j+2k+1\right)}H'_{23}\left(s_j+2k\right)\right)\\
K_{22} = {} & \frac{-H_{21}\left(s_j+2k+2\right)}{H^*\left(s_j+2k+2\right)}H'_{13}\left(s_j+2k+1\right)\frac{H'_{32}\left(s_j+2k\right)}{H_{22}\left(s_j+2k+1\right)}+\frac{H_{11}\left(s_j+2k+2\right)}{H^*\left(s_j+2k+2\right)}\left(\frac{H'_{23}\left(s_j+2k+1\right)H'_{32}\left(s_j+2k\right)}{H_{33}\left(s_j+2k+1\right)}-1\right)\\
K_{33} = {} & \frac{H'_{31}\left(s_j+2k+2\right)}{H_{33}\left(s_j+2k+2\right)}\left(H_{22}\left(s_j+2k+1\right)\frac{H'_{13}\left(s_j+2k\right)}{H^*\left(s_j+2k+1\right)}-\frac{H_{12}\left(s_j+2k+1\right)H'_{23}\left(s_j+2k\right)}{H^*\left(s_j+2k+1\right)}\right.\\
& \left.+H'_{32}\left(s_j+2k+2\right)\left(\frac{-H_{21}\left(s_j+2k+1\right)H'_{23}\left(s_j+2k\right)}{H^*\left(s_j+2k+1\right)}+\frac{H_{11}\left(s_j+2k+1\right)H'_{23}\left(s_j+2k\right)}{H^*\left(s_j+2k+1\right)}\right)-\Omega^*\tau_0\right)
\end{aligned}
\tag{A2}
$$

and

$$
\begin{aligned}
K'_{13} &= \frac{H_{22}\left(s_j+2k+1\right)}{H^*\left(s_j+2k+1\right)} H'_{13}\left(s_j+2k\right) - \frac{H_{11}\left(s_j+2k+2\right) H'_{23}\left(s_j+2k\right)}{H^*\left(s_j+2k+2\right)} \\
K'_{23} &= \frac{-H_{21}\left(s_j+2k+1\right) H'_{23}\left(s_j+2k\right)}{H^*\left(s_j+2k+1\right)} + \frac{H_{11}\left(s_j+2k+2\right) H'_{23}\left(s_j+2k\right)}{H^*\left(s_j+2k+1\right)} \\
K'_{31} &= \frac{H'_{31}\left(s_j+2k\right)}{H_{33}\left(s_j+2k+1\right)}, \quad K'_{32} = \frac{H'_{32}\left(s_j+2k\right)}{H_{33}\left(s_j+2k+1\right)}
\end{aligned}
\tag{A3}
$$