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Fixed Point of Presic Type Mapping in G-Metric Spaces

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, we give a fixed point theorem for Presic type contractive mapping in G-metric space. We also present an example to validate our result.

Keywords: G-metric space; fixed point; contractive mapping.

2010 Mathematics Subject Classification: 47H10, 54H25.

1 Introduction and Preliminaries

The common fixed point theory is very important and useful in Mathematics. It can be applied in various areas, for instant, variational inequalities, optimization, and approximation theory. The Banach contraction principle [1], is the simplest and one of the most versatile elementary results in

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fixed point theory. Recently Berinde introduced some new mappings that he called weak contraction mapping and almost contraction in a metric spaces [2–7]. He showed that Banach's, Kannan's [8], Chatterjea's [9] and Zamfirescu [10] mappings are almost contractions. In 2006, Mustafa and Sims [11] introduced a new structure called G-metric space as a generalization of the usual metric spaces. Afterwards based on the notion of a G-metric space, many fixed point results for different contractive conditions have been presented, for more details see [12–16].

Now, we mention briefly some fundamental definitions.

Definition 1.1. [11] Let X be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be function satisfying the following properties:

($G1$) $G(x, y, z) = 0$ if $x = y = z$;

($G2$) $0 < G(x, x, y)$; for all $x, y \in X$, with $x \neq y$;

($G3$) $G(x, x, y) \leq G(x, y, z)$, for all $x, y, z \in X$, with $z \neq y$;

($G4$) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables);

($G5$) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$, for all $x, y, z, a \in X$, (rectangle inequality).

Then the function G is called a generalized metric, or, more specifically a G-metric on X, and the pair (X, G) is called a G-metric space.

Definition 1.2. [11] Let (X, G) be a G-metric space. A sequence (x_n) in X is G-convergent to x $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$, if for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$.

Proposition 1.1. *[11] Let (X, G) be a G-metric space, then the following are equivalent.*

(i) (x_n) *is G-convergent to x.*

(ii) $G(x_n, x_n, x) \to 0$, *as* $n \to \infty$.

(iii) $G(x_n, x, x) \to 0$, *as* $n \to \infty$.

(iv) $G(x_m, x_n, x) \to 0$, *as* $n, m \to \infty$.

Definition 1.3. [11] Let (X, G) be a G-metric space, a sequence (x_n) is called G-Cauchy if for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$; that is, $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Proposition 1.2. *[11] If (X, G) is a G-metric space, then the following are equivalent.*

(i) *The sequence* (x_n) *is G-Cauchy.*

(ii) *For every* $\varepsilon > 0$,*there exists* $N \in \mathbb{N}$ *such that;* $G(x_n, x_m, x_m) < \varepsilon$, *for all* $n, m \geq N$.

Proposition 1.3. *[11] Let (X, G) be a G-metric space, then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Definition 1.4. [11] A G-metric space (X, G) is called symmetric if

$$G(x, y, y) = G(y, x, x) \quad \forall x, y \in X.$$

Definition 1.5. [11] Every G-metric space (X, G) defines a metric space (X, d_G) by

$$d_G(x, y) = G(x, y, y) + G(y, x, x)$$

for all $x, y \in X$. Note that if (X, G) is symmetric G-metric space then

$$d_G(x, y) = 2G(x, y, y) \quad \forall x, y \in X.$$

2 Main Results

Ćirić and Presic [17] showed Presic type generalization of the Banach contraction mapping in (X, d)-metric spaces. Also Dhasmana [18] showed fixed point theorem by using Presic type mapping in G-metric spaces. Further Gairola and Dhasmana [19] proved common fixed point theorems of Presic type in G-metric space which extends the result of Ćirić-Presic [17], Dhasmana [18] and George-Khan [20].

Considering the convergence of certain sequences S. B. Presic [21] generalized Banach contraction principle as follows:

Theorem 2.1. *Let (X, d) be a complete metric space, k a positive integer and $T : X^k \to X$ a mapping satisfying the following contractive type condition*

$$d(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_{k+1})) \leq q_1 d(x_1, x_2) + q_2 d(x_2, x_3) + ... + q_k d(x_k, x_{k+1}) \tag{2.1}$$

for every $x_1, x_2, ..., x_{k+1}$ in X where $q_1, q_2, ..., q_k$ are non negative constants such that $q_1 + q_2 + ... + q_k < 1$. Then there exist a point x in X such that $T(x, x, ..., x) = x$. Moreover, if $x_1, x_2, ..., x_k$, are arbitrary points in X and for $n \in N$,

$$x_{n+k} = T(x_n, x_{n+1}, ..., x_{n+k-1}) \quad (n = 1, 2, ...)$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and

$$\lim x_n = T(\lim x_n, \lim x_n, ..., \lim x_n).$$

Remark that condition (2.1) in the case $k = 1$ reduces to the well-known Banach contraction mapping principle. So, Theorem 2.1 is a generalization of the Banach fixed point theorem.

Ćirić and Presic [17] generalized Theorem 2.1 as follows:

Theorem 2.2. *([17]) Let (X, d) be a complete metric space, k a positive integer and $T : X^k \to X$ a mapping satisfying the following contractive type condition*

$$d(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_{k+1})) \leq \lambda \max_{1 \leq i \leq k} \{d(x_i, x_{i+1})\} \tag{2.2}$$

where $\lambda \in (0, 1)$ is constant and $x_1, x_2, ..., x_{k+1}$ in X. Then there exist a point x in X such that $T(x, x, ..., x) = x$. Moreover, if $x_1, x_2, ..., x_k$, are arbitrary points in X and for $n \in N$,

$$x_{n+k} = T(x_n, x_{n+1}, ..., x_{n+k-1}) \quad (n = 1, 2, ...)$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and

$$\lim x_n = T(\lim x_n, \lim x_n, ..., \lim x_n).$$

If in addition we suppose that on a diagonal $\triangle \subset X^k$

$$d(T(u, u, ..., u), T(v, v, ..., v)) < d(u, v) \tag{2.3}$$

holds for all $u, v \in X$, with $u \neq v$, then x is the unique point in X with $T(x, x, ..., x) = x$.

We will carry this idea in the Presic type almost contractive mapping in G-metric space.

Theorem 2.3. *Let (X, G) be complete G-metric space, $k \in Z^+$ and $T : X^k \to X$ a mapping satisfying the following contractive type condition*
$G(T(x_1, x_2, ..., x_k), T(y_1, y_2, ..., y_k), T(y_1, y_2, ..., y_k))$

$$\leq \ \delta \max\nolimits_{1 \leq i \leq k} \{G(x_i, y_i, y_i\} + L \min\{d_G(y_i, T(x_1, x_2, ..., x_k))\} \tag{2.4}$$

where $\delta \in (0,1)$ and $L \geq 0$, $x_1, x_2, ..., x_k$ and $y_1, y_2, ..., y_k$ are arbitrary points in X. Then there exist a point x in X such that $T(x, x, ..., x) = x$. Moreover, if $x_1, x_2, ..., x_k$ are arbitrary points in X and $n \in N$,

$$x_{n+k} = T(x_n, x_{n+1}, ..., x_{n+k-1}) \quad (n = 1, 2, ...) \tag{2.5}$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent and

$$\lim_{n\to\infty} x_n = T(\lim x_n, \lim x_n, ..., \lim x_n).$$

Thus x is a point in X with $T(x, x, ..., x) = x$.

Proof. $x_1, x_2, ..., x_k$, be k arbitrary points in X. Using these points define a sequence (x_n) as follows:

$$x_{n+k} = y_{n+k-1} = T(x_n, x_{n+1}, ..., x_{n+k-1}) \quad (n = 1, 2, ...) \tag{2.6}$$

For simplicity set $\gamma_n = G(x_n, y_n, y_n) = G(x_n, x_{n+1}, x_{n+1})$.
We shall prove by induction that for each $n \in \mathbb{N}$;

$$\gamma_n \leq M\theta^n \tag{2.7}$$

where $\theta = \delta^{\frac{1}{k}}$ and $M = \max\{\frac{\gamma_1}{\theta}, \frac{\gamma_2}{\theta^2}, ..., \frac{\gamma_k}{\theta^k}\}$.

According to the definition of M we see that (2.7) is true for $n = 1, 2, ..., k$.
Now let the following k inequalities:

$$\gamma_n \leq M\theta^n, \quad \gamma_{n+1} \leq M\theta^{n+1}, ..., \gamma_{n+k-1} \leq M\theta^{n+k-1}$$

be the induction hypotheses. Then we have:

$$\begin{aligned}\gamma_{n+k} &= G(x_{n+k}, x_{n+k+1}, x_{n+k+1})\\ &= G(T(x_n, x_{n+1}, ..., x_{n+k-1}), T(x_{n+1}, x_{n+2}, ..., x_{n+k}), T(x_{n+1}, x_{n+2}, ..., x_{n+k}))\end{aligned}$$

by (2.4) and the definition of γ_i; we obtain

$$\begin{aligned}\gamma_{n+k} =&G(T(x_n, x_{n+1}, ..., x_{n+k-1}), T(y_n, y_{n+1}, ..., y_{n+k-1}), T(y_n, y_{n+1}, ..., y_{n+k-1}))\\ \leq&\delta \max\{G(x_n, y_n, y_n), G(x_{n+1}, y_{n+1}, y_{n+1}), ..., G(x_{n+k-1}, y_{n+k-1}, y_{n+k-1})\}+\\ &L \min_{n\leq i\leq n+k-1}\{d_G(y_i, T(x_n, x_{n+1}, ..., x_{n+k-1}))\}\end{aligned}$$

here

$$x_{n+k} = y_{n+k-1} = T(x_n, x_{n+1}, ..., x_{n+k-1}) \quad (n = 1, 2, ...).$$

Then,

$$\begin{aligned}\gamma_{n+k} =&G(T(x_n, x_{n+1}, ..., x_{n+k-1}), T(y_n, y_{n+1}, ..., y_{n+k-1}), T(y_n, y_{n+1}, ..., y_{n+k-1}))\\ \leq&\delta \max\{G(x_n, y_n, y_n), G(x_{n+1}, y_{n+1}, y_{n+1}), ..., G(x_{n+k-1}, y_{n+k-1}, y_{n+k-1})\} + 0\\ =&\delta \max\{\gamma_n, \gamma_{n+1}, \gamma_{n+2}, ..., \gamma_{n+k-1}\}\\ \leq&\delta \max\{M\theta^n, M\theta^{n+1}, ..., M\theta^{n+k-1}\}\end{aligned}$$

as $\theta = \delta^{\frac{1}{k}}$

$$\begin{aligned}\gamma_{n+k} &\leq \delta M\theta^n \quad (\text{as } 0 \leq \theta \leq 1)\\ &= M\theta^{n+k}\end{aligned}$$

and the inductive proof of (2.7) is complete. Next using (2.7) for any $n, p \in \mathbb{N}$ we have the following argument:

$$\begin{aligned} G(x_n, x_{n+p}, x_{n+p}) \leq & G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + ... \\ & + G(x_{n+p-1}, x_{n+p}, x_{n+p}) \\ = & \gamma_n + \gamma_{n+1} + ... + \gamma_{n+p-1} \\ \leq & M\theta^n + M\theta^{n+1} + ... + M\theta^{n+p-1} \\ = & M\theta^n (\frac{1-\theta^p}{1-\theta}) \\ \leq & \frac{M\theta^n}{1-\theta} \end{aligned}$$

by which we conclude that (x_n) is a G-Cauchy sequence. Since (X, G) G-complete space, there exists x in X such that

$$\lim_{n\to\infty} x_n = x.$$

Then for any integer n we have

$$\begin{aligned} d_G(x_{n+k}, & T(x, x, ..., x)) = d_G(T(x, x, ..., x), T(x_n, x_{n+1}, ..., x_{n+k-1})) \\ \leq & d_G(T(x, x, ..., x), T(x, ..., x, x_n)) + d_G(T(x, ..., x, x_n), T(x, ..., x, x_n, x_{n+1})) + \\ & d_G(T(x, ..., x, x_n, x_{n+1}), T(x, ..., x, x_n, x_{n+1}, x_{n+2})) + ... + \\ & d_G(T(x, x_n, ..., x_{n+k-2}), T(x_n, x_{n+1}, ..., x_{n+k-1})) \\ = & G(T(x, x, ..., x), T(x, ..., x, x_n), T(x, ..., x, x_n)) + \\ & G(T(x, ..., x, x_n), T(x, ..., x, x_n, x_{n+1}), T(x, ..., x, x_n, x_{n+1})) + \\ & G(T(x, ..., x, x_n, x_{n+1}), T(x, ..., x, x_n, x_{n+1}, x_{n+2}), T(x, ..., x, x_n, x_{n+1}, x_{n+2})) \\ & + ... + G(T(x, x_n, ..., x_{n+k-2}), T(x_n, x_{n+1}, ..., x_{n+k-1}), T(x_n, x_{n+1}, ..., x_{n+k-1})) \\ \leq & \delta G(x, x_n, x_n) + L \min\{d_G(x, T(x, x, ..., x)), d_G(x_n, T(x, x, ..., x))\} \\ & + \delta \max\{G(x, x_n, x_n), G(x_n, x_{n+1}, x_{n+1})\} + L \min\{d_G(x, T(x, ..., x, x_n)), \\ & d_G(x_n, T(x, ..., x, x_n)), d_G(x_{n+1}, T(x, ..., x, x_n))\} + ... + \\ & \delta \max\{G(x, x_n, x_n), G(x_n, x_{n+1}, x_{n+1}), ..., G(x_{n+k-2}, x_{n+k-1}, x_{n+k-1})\} + \\ & L \min\{d_G(x_n, T(x, x_n, ..., x_{n+k-2})), d_G(x_{n+1}, T(x, x_n, ..., x_{n+k-2})), ..., \\ & d_G(x_{n+k-1}, T(x, x_n, x_{n+1}, ..., x_{n+k-2}))\}. \end{aligned}$$

Taking the limit when n tends to infinity we obtain

$$G(x, T(x, x, ..., x), T(x, x, ..., x)) \leq 0$$

which implies

$$T(x, x, ..., x) = x.$$

Thus we proved that;

$$\lim_{n\to\infty} x_n = T(\lim x_n, \lim x_n, ..., \lim x_n).$$

So, x is the a point in X. □

Example 2.4. *Let* $X = [0,3] \cup [4,8]$, *and* $T : X^3 \to X$ *be mapping defined by,*
$T(x,y,z) = \dfrac{x+y+z}{6}$, *if* $(x,y,z) \in [0,3] \times [0,3] \times [0,3]$.
$T(x,y,z) = 2 + \dfrac{x+y+z}{6}$, *if* $(x,y,z) \in [4,8] \times [4,8] \times [4,8]$.
$T(x,y,z) = \dfrac{x+y+z}{6} - \dfrac{1}{3}$, *if* $(x,y,z) \in [0,3] \times [0,3] \times [4,8]$ *or* $(x,y,z) \in [0,3] \times [4,8] \times [4,8]$ *or* $(x,y,z) \in [0,3] \times [4,8] \times [0,3]$ *or* $(x,y,z) \in [4,8] \times [4,8] \times [0,3]$ *or* $(x,y,z) \in [4,8] \times [0,3] \times [0,3]$ *or* $(x,y,z) \in [4,8] \times [0,3] \times [4,8]$ *and let*

$$G(x,y,z) = \max\{|x-y|, |y-z|, |x-z|\}.$$

Thus, we obtain

$$\begin{aligned} G(T(x,y,z), T(y,z,t), T(y,z,t)) = \max\{&|T(x,y,z) - T(y,z,t)|, |T(y,z,t) - T(y,z,t)|, \\ &|T(y,z,t) - T(x,y,z)|\} \end{aligned}$$

$$G(T(x,y,z), T(y,z,t), T(y,z,t)) = |T(x,y,z) - T(y,z,t)|$$

then,
for any $x, y, z \in [0,3]$ *we have* $T(x,y,z) = t \in [0,3]$,
for any $x, y, z \in [4,8]$ *we have* $T(x,y,z) = t \in [4,8]$.
Thus, for $x, y, z, t \in [0,3]$ *or* $x, y, z, t \in [4,8]$ *we have*

$$\begin{aligned} G(T(x,y,z), T(y,z,t), T(y,z,t)) =& \left|\frac{x+y+z}{6} - \frac{y+z+t}{6}\right| \\ \leq& \frac{1}{6}(|x-y| + |y-z| + |z-t|) \\ \leq& \frac{1}{2}\max\{G(x,y,y), G(y,z,z), G(z,t,t)\}. \end{aligned}$$

For $(x,y,z) \in [0,3]\times[0,3]\times[4,8]$ *or* $(x,y,z) \in [0,3]\times[4,8]\times[4,8]$ *or* $(x,y,z) \in [0,3]\times[4,8]\times[0,3]$ *or* $(x,y,z) \in [4,8]\times[4,8]\times[0,3]$ *or* $(x,y,z) \in [4,8]\times[0,3]\times[4,8]$ *we have* $T(x,y,z) = t \in [0,3]$ *we have*

$$\begin{aligned} G(T(x,y,z), T(y,z,t), T(y,z,t)) =& \left|\frac{x+y+z}{6} - \frac{1}{3} - \frac{y+z+t}{6} + \frac{1}{3}\right| \\ \leq& \frac{1}{6}(|x-y| + |y-z| + |z-t|) \\ \leq& \frac{1}{2}\max\{G(x,y,y), G(y,z,z), G(z,t,t)\}. \end{aligned}$$

For $(x,y,z) \in [4,8] \times [0,3] \times [0,3]$ *we have*

$$\begin{aligned} G(T(x,y,z), T(y,z,t), T(y,z,t)) =& \left|\frac{x+y+z}{6} - \frac{1}{3} - \frac{y+z+t}{6}\right| \\ \leq& \frac{1}{6}(|x-y| + |y-z| + |z-t|) \\ \leq& \frac{1}{2}\max\{G(x,y,y), G(y,z,z), G(z,t,t)\}. \end{aligned}$$

Thus for $L \geq 0$, T *satisfies (2.4) with* $\delta = \frac{1}{2}$ *but for* 0 *and* 4 *we have* $T(0,0,0) = 0$ *and* $T(4,4,4) = 4$.

3 Conclusion

Ćirić and Presic [17] showed Presic type generalization of the Banach contraction mapping in (X,d)-metric spaces. Also Dhasmana [18] showed fixed point theorem by using Presic type mapping in G-metric spaces. In this paper we moved this idea to Presic type almost contraction mapping in G-metric space. Our works generalizes several similar results in the literature.

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Competing Interests

Authors have declared that no competing interests exist.

References

[1] Banach S. Surles operations dansles ensembles abstracits et leur application aux equations integrales. Fund. Math. 1922;3:133-181.

[2] Berinde V. On the approximatation of weak contractive mappings. Carpathian J. Math. 2003;19(1):7-22.

[3] Berinde V. Approximatating fixed points of weak contractions using the Picard iteration. Nonlinear Anal. Forum. 2003;9(1):7-22.

[4] Berinde V. Approximatating fixed points of weak contractions using the Picard iteration. Nonlinear Anal. Forum. 2004;9(1):43-57.

[5] Berinde V. Approximatating of fixed points. Springer- Verlag, Berlin- Heidelberg; 2007.

[6] Berinde V. General constructive fixed point theorems for Ciric-type almost contractions in metric spaces. Carpathian J. 2008;24(2):10-19.

[7] Berinde V. Approximating fixed points of implicit almost contractions. Hacettepe J. Math. and Statistics. 2012;41(1):93-102.

[8] Kannan R. Some results on fixed points. Bull. Calcutta Math. Soc. 1968;60:71-76.

[9] Chatterjea SK. Fixed point theorems. C. R. Acad. Bulg. Sci. 1972;25:727-730.

[10] Zamfirescu T. Fixed point theorems in metric spaces. Arch. Math. (Basel). 1972;23:292-298.

[11] Mustafa Z, Sims B. A new approach to a generalized metric spaces. J. Nonlinear Convex Anal. 2006;7(2):289-297.

[12] Mustafa Z, Sims B. Some remarks concerninig Dmetric spaces. In Proceedings of the Internatinal Conferences on Fixed Point Theorey and Appl. 2003;189-198.

[13] Abbas M, Khan AR, Nazir T. Coupled common fixed point results in two generalized metric spaces. Appl. Math. and Comput. 2011;217(13):6328-6336.

[14] Abbas M, Nazir T, Doric D. Common fixed point of mappings satisfying (E.A) property in generalized metric spaces. Appl. Math. and Comput. 2012;218(14):7665-7670.

[15] Mustafa Z, Shatanawi W, Bataineh M. Existence of fixed point results in G-metric spaces. Int. J. Math. and Math. Sci. 2009;10. Article ID 283028.

[16] Ding HS, Karapınar E. A note on some coupled fixed point theorems on G-metric space. J. of Inequalities and Appl. 2012;2012. Article 170.
DOI: 10.1186/1029-242X-2012-170

[17] Ćirić LB, Presic SB. On Presic type generalization of the Banach contraction mapping principle. Acta Math. Univ. Comenianae. 2007;LXXVI(2):143-147.

[18] Dhasmana N. A fixed point theorem of Presic type in G-metric spaces. International J. Math. Arch. 2015;6(2):11-14.

[19] Gairola UC, Dhasmana N. A fixed point theorem of Presic type for a pair of maps in G-metric spaces. International J. Math. Arch. 2015;6(3):196-200.

[20] Gairola UC, Khan MS. On Presic type extension of Banach contraction principle. Int. J. Math. Anal. 2011;5(21):1019-1024.

[21] Presic SB. Sur une classe d'inequations aux difference finite et. sur la convergence de certains suites. Publl. de L'Inst. Math. Belgrade. 1965;5(19):75-78.

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