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Fuzzy Translations of Fuzzy Associative Ideals in BCI-algebras

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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Abstract

The aim of this paper is to introduce the concepts of fuzzy translation to fuzzy associative ideals in BCK/BCI-algebras. Also, the notion of fuzzy extensions and fuzzy multiplications of fuzzy associative ideals with several related properties are investigated. The relationships between fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy ideals are investigated.

Keywords: Fuzzy ideal; fuzzy associative ideal; fuzzy translation; fuzzy extension; fuzzy multiplication.

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1 Introduction

Non-classical logic has become a considerable formal tool for computer science and artificial intelligence to deal with fuzzy information and uncertainty information. Many-valued logic, a great extension and development of classical logic, has always been a crucial direction in non-classical logic. Since 1965 Zadeh's [1] invention, the concept of fuzzy sets has been extensively applied to many mathematical field. On the other hand, the theory of BCI/BCK-algebras introduced by

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Iseki [2] and it has been raised by Imai and Iseki [3]. Xi [4] applied the concept of fuzzy sets to BCK-algebras. In 2002, Liu and Zhang [5] introduced fuzzy associative ideals in BCI-algebras. In 2011, Jun [6] discussed fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy subalgebras and ideals in BCK/BCI-algebras. They investigated relations among fuzzy translations, fuzzy extensions and fuzzy multiplications. In 2015, Senapati et al., [7, 8], introduced the notation of fuzzy translation of fuzzy H-ideals and also they studied Intuitionistic fuzzy translation in BCI-algebras. Also, Senapati studied some applications of fuzzy translations in B-algebras [9, 10]. In 2016, Senapati et al., studied Atanassovs intuitionistic fuzzy bi-normed KU-ideals of a KU-algebra and they discussed some properties of it [11]. In this paper, fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy associative ideals in BCK/BCI-algebras are discussed. Relations among fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy associative ideals in BCK/BCI-algebras are also investigated.

2 Preliminaries

By a BCI-algebra we mean an algebra $(X; *, 0)$ of type $(2, 0)$ satisfying the following axioms:

(1) $((x * y) * (x * z)) * (z * y) = 0$,
(2) $(x * (x * y)) * y) = 0$,
(3) $x * x = 0$,
(4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

for all $x, y, z \in X$. We can define a partial ordering $'' \leq''$ on X by $x \leq y$ if and only if $x * y = 0$.

The following statements are true in any BCI-algebra X:

(1.1) $(x * y) * z = (x * z) * y$,
(1.2) $x * 0 = x$,
(1.3) $(x * z) * (y * z) \leq x * y$,
(1.4) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$,
(1.5) $0 * (x * y) = (0 * x) * (0 * y)$,
(1.6) $x * (x * (x * y)) = x * y$.

Definition 2.1. A non empty subset I of X is called an ideal of X if it satisfies:

(I_1) $0 \in I$,
(I_2) $x * y \in I$ and $y \in I$ imply $x \in I$.

Definition 2.2. ([12]) A nonempty subset I of X is called an associative ideal of X if it satisfies condition (I_1) and

(I_5) $(x * z) * (0 * y) \in I$ and $z \in I$ imply $y * x \in I$.

Definition 2.3. A fuzzy set μ of BCI-algebra X is called fuzzy ideal of X if it satisfies

(FI_1) $\mu(0) \geq \mu(x)$
(FI_2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$.

Definition 2.4. ([5]) A fuzzy set μ of BCI-algebra X is called fuzzy associative ideal of X if it satisfies (FI_1) and

(FI_4) $\mu(y * x) \geq \min\{\mu((x * z) * (0 * y)), \mu(z)\}$.

Proposition 2.1. *(see [5]) Let μ be a fuzzy set in a BCI-algebra X. Then μ is a fuzzy associative ideal of X if and only if for all $t \in [0, 1]$,*

$$\mu_t \neq \emptyset \Rightarrow \mu_t \ \text{ is an associative ideal of } X,$$

where $\mu_t = \{x \in X | \mu(x) \geq t\}$.

Definition 2.5. ([12]) Let μ and ν be fuzzy sets of a set X. The Cartesian product of μ and ν is defined by for all $x, y \in X$,

$$(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\}.$$

Remark 2.1. Throughout this paper, we take $\dagger = 1 - \sup\{\mu(x) | x \in X\}$ for any fuzzy set μ of X.

Definition 2.6. ([13]) Let μ be a fuzzy subset of X and let $\alpha \in [0, \dagger]$. A mapping $\mu^\dagger_\alpha : X \longrightarrow [0, 1]$ is called a fuzzy α-translation of μ if it satisfies $\mu^\dagger_\alpha(x) = \mu(x) + \alpha$ for all $x \in X$.

Proposition 2.2. *([7]) If the fuzzy α-translation $\mu^\dagger_\alpha$ of μ is an ideal of X, then it is order reversing.*

Definition 2.7. ([13]) Let μ_1 and μ_2 be fuzzy subsets of X. If $\mu_1 \leq \mu_2$, for all $x \in X$, then we say that μ_2 is a fuzzy extension of μ_1.

3 Major Section

Theorem 3.1. *If μ is a fuzzy associative ideal of X, then the fuzzy α-translation $\mu^\dagger_\alpha$ of μ is a fuzzy associative ideal of X, for all $\alpha \in [0, \dagger]$.*

Proof. Assume that μ is a fuzzy associative ideal of X and let $\alpha \in [0, \dagger]$. Then we have

$$\mu^\dagger_\alpha(0) = \mu(0) + \alpha \geq \mu(x) + \alpha = \mu^\dagger_\alpha(x),$$

and for all $x, y, z \in X$ we have

$$\begin{aligned}\mu^\dagger_\alpha(y * x) &= \mu(y * x) + \alpha \\ &\geq \min\{\mu((x * z) * (0 * y)), \mu(z)\} + \alpha \\ &= \min\{\mu((x * z) * (0 * y)) + \alpha, \mu(z) + \alpha\} \\ &= \min\{\mu^\dagger_\alpha((x * z) * (0 * y)), \mu^\dagger_\alpha(z)\}.\end{aligned}$$

Hence, the fuzzy α-translation $\mu^\dagger_\alpha$ of μ is a fuzzy associative ideal of X. $\square$

Theorem 3.2. *Let μ be a fuzzy subset of X such that the fuzzy α-translation $\mu^\dagger_\alpha$ of μ is a fuzzy associative ideal of X for some $\alpha \in [0, \dagger]$. Then, μ is a fuzzy associative ideal of X.*

Proof. Assume that $\mu^\dagger_\alpha$ is a fuzzy associative ideal of X for some $\alpha \in [0, \dagger]$. Let $x \in X$, then

$$\mu(0) + \alpha = \mu^\dagger_\alpha(0) \geq \mu^\dagger_\alpha(x) = \mu(x) + \alpha,$$

so $\mu(0) \geq \mu(x)$. Also, for all $x, y, z \in X$, we have

$$\begin{aligned}\mu(y * x) + \alpha &= \mu^\dagger_\alpha(y * x) \\ &\geq \min\{\mu^\dagger_\alpha((x * z) * (0 * y)), \mu^\dagger_\alpha(z)\} \\ &= \min\{\mu((x * z) * (0 * y)) + \alpha, \mu(z) + \alpha\} \\ &= \min\{\mu((x * z) * (0 * y)), \mu(z)\} + \alpha,\end{aligned}$$

so, $\mu(y * x) \geq \min\{\mu((x * z) * (0 * y)), \mu(z)\}$. Therefore μ is a fuzzy associative ideal of X. $\square$

Theorem 3.3. *If the fuzzy α-translation $\mu^\dagger_\alpha$ of μ is a fuzzy associative ideal of X, for all $\alpha \in [0, \dagger]$ then it is a fuzzy subalgebra of X.*

Proof. Let the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ is a fuzzy associative ideal of X. Then, we have $\mu^{\dagger}_{\alpha}(y*x) \geq \min\{\mu^{\dagger}_{\alpha}((x*z)*(0*y)), \mu^{\dagger}_{\alpha}(z)\}$. Since by [5], μ is a subalgebra, we have

$$\begin{aligned}\mu^{\dagger}_{\alpha}(y*x) &= \mu(y*x)+\alpha\\ &\geq \min\{\mu(x),\mu(y)\}+\alpha\\ &= \min\{\mu(x)+\alpha,\mu(y)+\alpha\}\\ &= \min\{\mu^{\dagger}_{\alpha}(x),\mu^{\dagger}_{\alpha}(y)\}.\end{aligned}$$

Therefore, $\mu^{\dagger}_{\alpha}$ is a fuzzy subalgebra of X.

□

The converse of Theorem 3.3, by example 3.4 of [12], is not true.

Theorem 3.4. *Let μ be a fuzzy subset of X such that the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ is a fuzzy associative ideal of X for $\alpha \in [0, \dagger]$. Then the set $I_\mu = \{x \in X | \mu^{\dagger}_{\alpha}(x) = \mu^{\dagger}_{\alpha}(0)\}$ is an associative ideal of X.*

Proof. Clearly, $0 \in I_\mu$. Assume that $x, y, z \in X$ such that $(x*z)*(0*y), z \in I_\mu$, then

$$\mu^{\dagger}_{\alpha}((x*z)*(0*y)) = \mu^{\dagger}_{\alpha}(0) = \mu^{\dagger}_{\alpha}(z).$$

Thus, we have

$$\mu^{\dagger}_{\alpha}(y*x) \geq \min\{\mu^{\dagger}_{\alpha}((x*z)*(0*y)), \mu^{\dagger}_{\alpha}(z)\} = \mu^{\dagger}_{\alpha}(0).$$

Since, $\mu^{\dagger}_{\alpha}$ of μ is a fuzzy associative ideal of X, we conclude that $\mu^{\dagger}_{\alpha}(y*x) = \mu^{\dagger}_{\alpha}(0)$. Therefore $\mu(y*x)+\alpha = \mu(0)+\alpha$, i.e., $\mu(y*x) = \mu(0)$, so that $y*x \in I_\mu$. Therefore, I_μ is an associative ideal of X.

□

Theorem 3.5. *Let μ be a fuzzy subset of X such that the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ is a fuzzy ideal of X, then the following statements are equivalent:*

(i) $\mu^{\dagger}_{\alpha}$ *is a fuzzy associative ideal of* X,
(ii) $\mu^{\dagger}_{\alpha}((x*z)*(0*y)) \leq \mu^{\dagger}_{\alpha}(y*(x*z))$,
(iii) $\mu^{\dagger}_{\alpha}(x*(0*y)) \leq \mu^{\dagger}_{\alpha}(y*x)$.

Proof. $(i) \Rightarrow (ii)$: Let $\mu^{\dagger}_{\alpha}$ is a fuzzy associative ideal of X. Then, for all $x, y \in X$ we have

$$\mu^{\dagger}_{\alpha}(y*(x*z)) \geq \min\{\mu^{\dagger}_{\alpha}(((x*z)*0)*(0*y)), \mu^{\dagger}_{\alpha}(0)\} = \mu^{\dagger}_{\alpha}((x*z)*(0*y)).$$

$(ii) \Rightarrow (iii)$: Taking $z = 0$ in (ii), induces (iii).
$(iii) \Rightarrow (i)$: Assume that $x, y, z \in X$. We have

$$(x*(0*y))*((x*z)*(0*y)) \leq x*(x*z) \leq z,$$

so,

$$\mu^{\dagger}_{\alpha}(x*(0*y)) \geq \min\{\mu^{\dagger}_{\alpha}((x*z)*(0*y)), \mu^{\dagger}_{\alpha}(z)\},$$

by Corollary 3.6 of [6]. Now, by part (iii),

$$\mu^{\dagger}_{\alpha}(y*x) \geq \min\{\mu^{\dagger}_{\alpha}((x*z)*(0*y)), \mu^{\dagger}_{\alpha}(z)\}.$$

Hence $\mu^{\dagger}_{\alpha}$ is fuzzy associative ideal of X.

□

Theorem 3.6. *Let μ be a fuzzy subset of X such that the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ is a fuzzy ideal of X. If for all $x, y, z \in X$, $\mu^{\dagger}_{\alpha}((x*y)*z) = \mu^{\dagger}_{\alpha}(x*(y*z))$, then $\mu^{\dagger}_{\alpha}$ is a fuzzy associative ideal of X.*

Proof. For all $x \in X$ we have $\mu^{\dagger}_{\alpha}(0*x) = \mu^{\dagger}_{\alpha}((x*x)*x) = \mu^{\dagger}_{\alpha}(x*(x*x)) = \mu^{\dagger}_{\alpha}(x)$, i.e., $\mu^{\dagger}_{\alpha}(0*x) = \mu^{\dagger}_{\alpha}(x)$. Now, for all $x, y \in X$ we have

$$
\begin{aligned}
\mu^{\dagger}_{\alpha}(x*(0*y)) = \mu^{\dagger}_{\alpha}(x*y) & \\
&= \mu^{\dagger}_{\alpha}(0*(x*y)) \\
&= \mu^{\dagger}_{\alpha}((0*x)*y) \\
&= \mu^{\dagger}_{\alpha}((0*y)*x) \\
&= \mu^{\dagger}_{\alpha}(0*(y*x)) \\
&= \mu^{\dagger}_{\alpha}(0*(y*x)) \\
&= \mu^{\dagger}_{\alpha}(y*x).
\end{aligned}
$$

Hence by Theore 3.5(iii), $\mu^{\dagger}_{\alpha}$ is a fuzzy associative ideal of X. □

Definition 3.1. Let μ_1 and μ_2 be fuzzy subsets of X. Then μ_2 is called a fuzzy associative ideal extension of μ_1 if the following statements are valid:

(i) μ_2 is a fuzzy extension of μ_1,
(ii) If μ_1 is a fuzzy associative ideal of X, then μ_2 is a fuzzy associative ideal of X.

Theorem 3.7. *Let μ be a fuzzy associative ideal of X and $\alpha \in [0, \dagger]$. Then, the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ is a fuzzy associative ideal extension of μ.*

Proof. It's clear from the definition of fuzzy α-translation. □

The following example show that a fuzzy associative ideal extension of a fuzzy associative ideal μ may not be represented as a fuzzy α-translation of μ:

Example 3.8. *Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table:*

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let μ be a fuzzy subset of X defined by:

X	0	a	b	c
μ	0.6	0.6	0.4	0.4

Then, μ is a fuzzy associative ideal of X. Let ν be a fuzzy subset of X given by:

X	0	a	b	c
ν	0.63	0.63	0.41	0.41

Then, ν is a fuzzy associative ideal extension of μ. But it is not the fuzzy α-translation $\mu^{\dagger}_{\alpha}$ of μ, for all $\alpha \in [0, \dagger]$.

For a fuzzy subset μ of X, $\alpha \in [0, \dagger]$ and $t \in [0, 1]$ with $t \geq \alpha$, let

$$U_{\alpha}(\mu; t) := \{x \in X | \mu(x) \geq t - \alpha\}.$$

It is easy to check that, if μ is a fuzzy a-ideal of X, then for all $t \in Im(\mu)$ with $t \geq \alpha$, $U_{\alpha}(\mu; t)$ is an a-ideal of X. The following example shows that, if we do not give a condition that μ is a fuzzy a-ideal of X, then $U_{\alpha}(\mu; t)$ is not an a-ideal of X:

Example 3.9. *Consider a BCI-algebra $X = \{0, 1, 2\}$ with the following Cayley table:*

$*$	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Let μ be a fuzzy subset of X defined by:

X	0	1	2
μ	0.6	0.4	0.4

Then, μ is not a fuzzy associative ideal of X, because

$$\mu(2 * 1) = \mu(1) = 0.4 \not\geq 0.6 = \min\{\mu((1 * 0) * (0 * 2)), \mu(0)\}.$$

Now, for $\alpha = 0.2$ and $t = 0.62$, we have $U_\alpha(\mu; t) = \{0\}$ which is not an a-ideal of X, because $(1 * 0) * (0 * 2) = 0 \in U_\alpha(\mu; t)$, but $2 * 1 = 1 \notin U_\alpha(\mu; t)$.

Theorem 3.10. *Let μ be a fuzzy subset of X and $\alpha \in [0, \dagger]$. Then, the fuzzy α-translation $\mu^\dagger_\alpha$ of μ is a fuzzy associative ideal of X if and only if $U_\alpha(\mu; t)$ is an associative ideal of X, for all $t \in Im(\mu)$ with $t > \alpha$.*

Proof. Suppose that $\mu^\dagger_\alpha$ is a fuzzy associative ideal of X and $t \in Im(\mu)$ with $t > \alpha$. Since $\mu^\dagger_\alpha(0) \geq \mu^\dagger_\alpha(x)$, for all $x \in X$, we have $\mu(0) + \alpha = \mu^\dagger_\alpha(0) \geq \mu^\dagger_\alpha(x) = \mu(x) + \alpha \geq t$, for $x \in U_\alpha(\mu; t)$, so $0 \in U_\alpha(\mu; t)$. Let $x, y, z \in X$ such that $(x * z) * (0 * y), z \in U_\alpha(\mu; t)$, then

$$\mu((x * z) * (0 * y)) \geq t - \alpha \quad , \quad \mu(z) \geq t - \alpha$$

i.e.,

$$\mu^\dagger_\alpha((x * z) * (0 * y)) \geq t \quad , \quad \mu^\dagger_\alpha(z) \geq t.$$

Since$\mu^\dagger_\alpha$ is a fuzzy associative ideal. So, we have

$$\mu(y * x) + \alpha = \mu^\dagger_\alpha(y * x) \geq \min\{\mu^\dagger_\alpha((x * z) * (0 * y)), \mu^\dagger_\alpha(z)\} \geq t,$$

that is, $\mu(y * x) \geq t - \alpha$ so that $y * x \in U_\alpha(\mu; t)$. Therefore, $U_\alpha(\mu; t)$ is an associative ideal of X.

Conversely, suppose that for all $t \in Im(\mu)$ with $t > \alpha$, $U_\alpha(\mu; t)$ is an associative ideal of X. If there exists $a \in X$ such that $\mu^\dagger_\alpha(0) < \beta \leq \mu^\dagger_\alpha(a)$, then $\mu(a) \geq \beta - \alpha$ but $\mu(0) < \beta - \alpha$. Therefore $a \in U_\alpha(\mu; t)$ and $0 \notin U_\alpha(\mu; t)$. Hence it's contradiction and so for all $x \in X$, $\mu^\dagger_\alpha(0) \geq \mu^\dagger_\alpha(x)$. Now assume that there exist $a, b, c \in X$ such that $\mu^\dagger_\alpha(b * a) < \gamma \leq \min\{\mu^\dagger_\alpha((a * c) * (0 * b)), \mu^\dagger_\alpha(c)\}$. Then $\mu((a*c)*(0*b)) \geq \gamma - \alpha$ and $\mu(z) \geq \gamma - \alpha$ but $\mu(b*a) < \gamma - \alpha$. Therefore $(a*c)*(0*b), c \in U_\alpha(\mu; t)$ but $b * a \notin U_\alpha(\mu; t)$, which is a contradiction. Hence, $\mu^\dagger_\alpha$ is a fuzzy associative ideal of X. □

Clearly, the following theorem is holds by definition 2.4.

Theorem 3.11. *Let μ be a fuzzy associative ideal of X and let $\alpha, \beta \in [0, \dagger]$. If $\alpha \geq \beta$ then the fuzzy α-translation $\mu^\dagger_\alpha$ of μ is a fuzzy associative ideal extension of the fuzzy β-translation $\mu^\dagger_\beta$ of μ.*

Theorem 3.12. *Let μ be a fuzzy associative ideal of X and $\beta \in [0, \dagger]$. For every fuzzy associative ideal extension ν of the fuzzy β-translation $\mu^\dagger_\beta$ of μ, there exists $\alpha \in [0, \dagger]$ such that $\alpha \geq \beta$ and ν is a fuzzy associative ideal extension of the fuzzy α-translation $\mu^\dagger_\alpha$ of μ.*

Proof. For every fuzzy associative ideal μ of X and $\beta \in [0, \dagger]$, the fuzzy β-translation $\mu^\dagger_\beta$ of μ is a fuzzy associative ideal of X. Now, if μ is a fuzzy associative ideal extension of $\mu^\dagger_\beta$, then there exists $\alpha \in [0, \dagger]$ such that $\alpha \geq \beta$ and for all $x \in X$, $\nu(x) \geq \mu^\dagger_\alpha$. □

Definition 3.2. Let μ be a fuzzy subset of X and $\gamma \in [0,1]$. A fuzzy γ-multiplication of μ, denoted by μ_γ^m, is defined to be a mapping $\mu_\gamma^m : X \longrightarrow [0,1]$, by $\mu_\gamma^m(x) = \mu(x) \cdot \gamma$.

Now, we have the following theorem and it is holds from Definition 3.2:

Theorem 3.13. *If μ is a fuzzy associative ideal of X, then the fuzzy γ-multiplication of μ is a fuzzy associative ideal of X, for all $\gamma \in [0,1]$.*

Theorem 3.14. *Let μ be a fuzzy subset of X. Then μ is a fuzzy associative ideal of X if and only if the fuzzy γ-multiplication μ_γ^m of μ is a fuzzy associative ideal of X, for all $\gamma \in [0,1]$.*

Proof. ($\Rightarrow$) By Theorem 3.13, is clear.
($\Leftarrow$) Assume that μ_γ^m of μ is a fuzzy associative ideal of X, for all $\gamma \in [0,1]$. Thus,

$$\mu(0) \cdot \gamma = \mu_\gamma^m(0) \geq \mu_\gamma^m(x) = \mu(x) \cdot \gamma,$$

i.e., for all $x \in X$, $\mu(0) \geq \mu(x)$. Also, for $x, y, z \in X$, we have

$$\begin{aligned}\mu(y * x) \cdot \gamma &= \mu_\gamma^m(y * x)\\ &\geq \min\{\mu_\gamma^m((x * z) * (0 * y)), \mu_\gamma^m(z)\}\\ &= \min\{\mu((x * z) * (0 * y)) \cdot \gamma, \mu(z) \cdot \gamma\}\\ &= \min\{\mu((x * z) * (0 * y)), \mu(z)\} \cdot \gamma,\end{aligned}$$

which implies that $\mu(y * x) \geq \min\{\mu((x * z) * (0 * y)), \mu(z)\}$. Therefore μ is a fuzzy associative ideal of X. □

Theorem 3.15. *Let μ be a fuzzy subset of X, $\alpha \in [0, \dagger]$ and $\gamma \in (0,1]$. Then, every fuzzy α-translation $\mu_\alpha^\dagger$ of μ is a fuzzy associative ideal extension of the fuzzy γ-multiplication μ_γ^m of μ.*

Proof. For all $x \in X$, we have

$$\mu_\alpha^\dagger(x) = \mu(x) + \alpha \geq \mu(x) > \mu(x) \cdot \gamma = \mu_\gamma^m(x)$$

and so $\mu_\alpha^\dagger$ is a fuzzy extension of μ_γ^m. Assume that μ_γ^m is a fuzzy associative ideal of X. Then, by Theorem 3.14, μ is a fuzzy associative ideal of X. It follows from Theorem 3.1 that the fuzzy α-translation $\mu_\alpha^\dagger$ of μ is a fuzzy associative ideal of X, for all $\alpha \in [0, \dagger]$. Therefore, every fuzzy α-translation $\mu_\alpha^\dagger$ of μ is a fuzzy associative ideal extension of fuzzy γ-multiplication μ_γ^m of μ. □

Theorem 3.15 for $\gamma = 0$, by the following example is not holds:

Example 3.16. *Let $(\mathbb{Z}, *, 0)$ be a BCI-algebra, where $\mathbb{Z}$ is the set of all integers we define that the operation $*$ by, $x * y := x - y$, for all $x, y \in \mathbb{Z}$. Define a fuzzy subset $\mu : \mathbb{Z} \longrightarrow [0,1]$, by*

$$\mu(x) = \begin{cases} t_0 & ; \quad x > 2 \\ t_1 & ; \quad x \leq 2, \end{cases}$$

where $t_0, t_1 \in [0,1]$ and $t_1 > t_0$. If $\gamma = 0$, then for all $x, y, z \in \mathbb{Z}$ we have

$$\mu_0^m(y * x) = 0 = \min\{\mu_0^m((x * z) * (0 * y)), \mu_0^m(z)\},$$

i.e., μ_0^m is a fuzzy a-ideal of $\mathbb{Z}$. Now, by taking $x = 0, y = 3, z = 1$, we have

$$\begin{aligned}\mu_\alpha^\dagger(3 * 0) = \mu_\alpha^\dagger(3) &= 0.4 + \alpha\\ &< 0.6 + \alpha\\ &= \min\{\mu((0 * 1) * (0 * 3)), \mu(1)\} + \alpha\\ &= \min\{\mu((0 * 1) * (0 * 3)) + \alpha, \mu(1) + \alpha\}\\ &= \min\{\mu_\alpha^\dagger((0 * 1) * (0 * 3)), \mu_\alpha^\dagger(1)\},\end{aligned}$$

for all $\alpha \in [0, t_0]$, which shows that $\mu_\alpha^\dagger$ is not a fuzzy a-ideal of $\mathbb{Z}$. Hence, $\mu_\alpha^\dagger$ is not a fuzzy a-ideal extension of μ_0^m, for all $\alpha \in [0, t_0]$ (see also [7]).

By easy calculation, we can show that the intersection of fuzzy a-ideal extensions of a fuzzy subset μ of a BCI-algebra X is a fuzzy a-ideal extension of μ. The following example show that the union of fuzzy a-ideal extensions of a fuzzy subset μ of X is not a fuzzy a-ideal extension of μ.

Example 3.17. *Consider a BCI-algebra $X = \{0, a, b, c\}$ with the following Cayley table:*

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let μ be a fuzzy subset of X defined by:

X	0	a	b	c
μ	0.6	0.6	0.4	0.4

Then, μ is a fuzzy associative ideal of X. Let ν and η be fuzzy subsets of X given by:

X	0	a	b	c
ν	0.63	0.62	0.52	0.41
η	0.63	0.51	0.62	0.42

Then, ν and η are fuzzy associative ideal extension of μ. The union $\nu \cup \eta$ is a fuzzy extension of μ, but it is not a fuzzy a-ideal extension of μ, because

$$(\nu \cup \eta)(a * b) = 0.42 \ngeq \min\{(\nu \cup \eta)((b * b) * (0 * a)), (\nu \cup \eta)(b)\} = 0.62.$$

Theorem 3.18. *Let μ and ν be two fuzzy a-ideals of BCI-algebra X and for all $\alpha \in [0, \dagger]$, $\mu_\alpha^\dagger$ and $\nu_\alpha^\dagger$ be the fuzzy α-translation of μ and ν, respectively. Then $\mu_\alpha^\dagger \times \nu_\alpha^\dagger$ is a fuzzy a-ideal of $X \times X$, for all $\alpha \in [0, \dagger]$.*

Proof. For all $\bar{x} = (x_1, x_2) \in X \times X$, we obtain:

$$\begin{aligned}
(\mu_\alpha^\dagger \times \nu_\alpha^\dagger)(\bar{0}) &= (\mu_\alpha^\dagger \times \nu_\alpha^\dagger)(0, 0) \\
&= \min\{\mu_\alpha^\dagger(0), \nu_\alpha^\dagger(0)\} \\
&\geq \min\{\mu_\alpha^\dagger(x_1), \nu_\alpha^\dagger(x_2)\} \\
&= (\mu_\alpha^\dagger \times \nu_\alpha^\dagger)(x_1, x_2) \\
&= (\mu_\alpha^\dagger \times \nu_\alpha^\dagger)(\bar{x}).
\end{aligned}$$

Also, for all $\bar{x} = (x_1, x_2), \bar{y} = (y_1, y_2)$ and $\bar{z} = (z_1, z_2)$ in $X \times X$, we have:

$$\begin{aligned}
&(\mu_\alpha^\dagger \times \nu_\alpha^\dagger)(\bar{y} * \bar{x}) = (\mu_\alpha^\dagger \times \nu_\alpha^\dagger)(y_1 * x_1, y_2 * x_2) \\
&= \min\{\mu_\alpha^\dagger(y_1 * x_1), \nu_\alpha^\dagger(y_2 * x_2)\} \\
&\geq \min\{\min\{\mu_\alpha^\dagger((x_1 * z_1) * (0 * y_1)), \mu_\alpha^\dagger(z_1)\}, \min\{\nu_\alpha^\dagger((x_1 * z_1) * (0 * y_1)), \nu_\alpha^\dagger(z_1)\}\} \\
&= \min\{\min\{\mu_\alpha^\dagger((x_1 * z_1) * (0 * y_1)), \nu_\alpha^\dagger((x_1 * z_1) * (0 * y_1))\}, \min\{\mu_\alpha^\dagger(z_1), \nu_\alpha^\dagger(z_1)\}\} \\
&= \min\{(\mu_\alpha^\dagger \times \nu_\alpha^\dagger)(((x_1, x_2) * (z_1, z_2)) * ((0, 0) * (y_1, y_2))), (\mu_\alpha^\dagger \times \nu_\alpha^\dagger)(z_1, z_2)\} \\
&= \min\{(\mu_\alpha^\dagger \times \nu_\alpha^\dagger)((\bar{x} * \bar{z}) * (\bar{0} * \bar{y})), (\mu_\alpha^\dagger \times \nu_\alpha^\dagger)(\bar{z})\}.
\end{aligned}$$

Therefore, $\mu_\alpha^\dagger \times \nu_\alpha^\dagger$ is a fuzzy a-ideal of $X \times X$. □

4 Conclusions

We introduced the notion of a fuzzy translation of fuzzy associative ideal as a generalization of a fuzzy associative ideal of a BCI-algebra. Relations between fuzzy translation of fuzzy ideals and fuzzy translation of fuzzy associative ideals are given. Conditions for a fuzzy translation of fuzzy ideal to be a fuzzy translation of fuzzy associative ideal are provided. For future works, we can work on some other fuzzy ideals in BCI-algebra, for example, fuzzy implicative ideals, fuzzy positive implicative ideals, ... Also, we can find some relations between fuzzy translation of fuzzy associative ideal and other fuzzy ideals such as fuzzy implicative, positive implicative ideals, p-ideal, H-ideals.

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Competing Interests

Author has declared that no competing interests exist.

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